35-th International Mathematical Olympiad

Hong Kong, July 9–22, 1994

First Day – July 13

1. Let m and n be positive integers. The set $A = \{a_1, a_2, \ldots, a_m\}$ is a subset of $1, 2, \ldots, n$. Whenever $a_i + a_j \leq n$, $1 \leq i \leq j \leq m$, $a_i + a_j$ also belongs to A. Prove that
$$\frac{a_1 + a_2 + \cdots + a_m}{m} \geq \frac{n+1}{2}.$$
(France)

2. N is an arbitrary point on the bisector of $\angle BAC$. P and O are points on the lines AB and AN, respectively, such that $\measuredangle ANP = 90^\circ = \measuredangle APO$. Q is an arbitrary point on NP, and an arbitrary line through Q meets the lines AB and AC at E and F respectively. Prove that $\measuredangle OQE = 90^\circ$ if and only if $QE = QF$. *(Armenia/Australia)*

3. For any positive integer k, A_k is the subset of $\{k+1, k+2, \ldots, 2k\}$ consisting of all elements whose digits in base 2 contain exactly three 1's. Let $f(k)$ denote the number of elements in A_k.
   (a) Prove that for any positive integer m, $f(k) = m$ has at least one solution.
   (b) Determine all positive integers m for which $f(k) = m$ has a unique solution. *(Romania)*

Second Day – July 14

4. Determine all pairs (m, n) of positive integers such that $\dfrac{n^3+1}{mn-1}$ is an integer. *(Australia)*

5. Let S be the set of real numbers greater than -1. Find all functions $f : S \to S$ such that
$$f(x + f(y) + xf(y)) = y + f(x) + yf(x) \quad \text{for all } x \text{ and } y \text{ in } S,$$
and $f(x)/x$ is strictly increasing for $-1 < x < 0$ and for $0 < x$. *(Great Britain)*

6. Find a set A of positive integers such that for any infinite set P of prime numbers, there exist positive integers $m \in A$ and $n \notin A$, both the product of the same number (at least two) of distinct elements of P. *(Finland)*