

12-th Balkan Mathematical Olympiad

Plovdiv, Bulgaria – May 9, 1995

1. Define $x * y = \frac{x + y}{1 + xy}$. Evaluate $(\dots(((2 * 3) * 4) * 5) * \dots) * 1995$.

(FYR Macedonia)

2. Circles $c_1(O_1, r_1)$ and $c_2(O_2, r_2)$, $r_2 > r_1$, intersect at A and B so that $\angle O_1AO_2 = 90^\circ$. The line O_1O_2 meets c_1 at C and D , and c_2 at E and F (in the order $C - E - D - F$). The line BE meets c_1 at K and AC at M , and the line BD meets c_2 at L and AF at N . Prove that

$$\frac{r_2}{r_1} = \frac{KE}{KM} \cdot \frac{LN}{LD}. \quad (\text{Greece})$$

3. Let a and b be natural numbers with $a > b$ and $2 \mid a + b$. Prove that the solutions of the equation

$$x^2 - (a^2 - a + 1)(x - b^2 - 1) - (b^2 + 1)^2 = 0$$

are natural numbers, none of which is a perfect square. (Albania)

4. Let n be a natural number and S be the set of points (x, y) with $x, y \in \{1, 2, \dots, n\}$. Let T be the set of all squares with the vertices in the set S . We denote by a_k ($k \geq 0$) the number of (unordered) pairs of points for which there are exactly k squares in T having these two points as vertices. Show that $a_0 = a_2 + 2a_3$. (Yugoslavia)