37-th International Mathematical Olympiad

Mumbai, India, July 5–17, 1996

First Day – July 10

1. We are given a positive integer r and a rectangular board $ABCD$ with dimensions $|AB| = 20$, $|BC| = 12$. The rectangle is divided into a grid of 20×12 unit squares. The following moves are permitted on the board: One can move from one square to another only if the distance between the centers of the two squares is $\sqrt{r}$. The task is to find a sequence of moves leading from the square corresponding to vertex A to the square corresponding to vertex B.

   (a) Show that the task cannot be done if r is divisible by 2 or 3.

   (b) Prove that the task is possible when $r = 73$.

   (c) Is there a solution when $r = 97$? *(Finland)*

2. Let P be a point inside $\triangle ABC$ such that

$$\angle APB - \angle C = \angle APC - \angle B.$$

   Let D, E be the incenters of $\triangle APB, \triangle APC$ respectively. Show that AP, BD, and CE meet in a point. *(Canada)*

3. Let $\mathbb{N}_0$ denote the set of nonnegative integers. Find all functions f from $\mathbb{N}_0$ into itself such that

$$f(m + f(n)) = f(f(m)) + f(n), \qquad \forall m, n \in \mathbb{N}_0.$$ *(Romania)*

Second Day – July 11

4. The positive integers a and b are such that the numbers $15a + 16b$ and $16a - 15b$ are both squares of positive integers. What is the least possible value that can be taken on by the smaller of these two squares? *(Russia)*

5. Let $ABCDEF$ be a convex hexagon such that AB is parallel to DE, BC is parallel to EF, and CD is parallel to AF. Let R_A, R_C, R_E be the circumradii of triangles FAB, BCD, DEF respectively, and let P denote the perimeter of the hexagon. Prove that

$$R_A + R_C + R_E \geq \frac{P}{2}.$$ *(Armenia)*

6. Let p, q, n be three positive integers with $p + q < n$. Let $(x_0, x_1, \ldots, x_n)$ be an $(n+1)$-tuple of integers satisfying the following conditions:

   (i) $x_0 = x_n = 0$.

   (ii) For each i with $1 \leq i \leq n$, either $x_i - x_{i-1} = p$ or $x_i - x_{i-1} = -q$.

   Show that there exists a pair (i, j) of distinct indices with $(i, j) \neq (0, n)$ such that $x_i = x_j$. *(France)*