Yugoslav IMO Team Selection Test 1999

Belgrade, May 22, 1999

Time allowed 4 *hours.*
Each problem is worth 25 *points.*

1. For a natural number n, let $P(x)$ be the polynomial of $2n-$th degree such that
$$P(0) = 1 \quad \text{and} \quad P(k) = 2^{k-1} \text{ for } k = 1, 2, \dots, 2n.$$
Prove that $2P(2n+1) - P(2n+2) = 1$.

2. Let ABC be a triangle with $\angle A = 90^\circ$ and $\angle B < \angle C$. The tangent at A to the circumcircle k of $\triangle ABC$ meets line BC at D. Let E be the reflection of A in BC, X be the foot of the perpendicular from A to BE, and Y be the midpoint of segment AX. Line BY intersects k again at Z. Prove that line BD is tangent to the circumcircle of $\triangle ADZ$.

3. Consider the set $A_n = \{x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n\}$ of $2n$ variables. How many permutations of set A_n are there for which it is possible to assign real values from the interval $(0, 1)$ to the $2n$ variables so that:

   (i) $x_i + y_i = 1$ for each i;

   (ii) $x_1 < x_2 < \cdots < x_n$;

   (iii) the $2n$ terms of the permutation form a strictly increasing sequence?

4. For a natural number d, M_d denotes the set of natural numbers which are not representable as the sum of at least two consecutive terms of an arithmetic progression with the common difference d whose terms are integers. Prove that each $c \in M_3$ can be written in the form $c = ab$, where $a \in M_1$ and $b \in M_2 \setminus \{2\}$.