

# 18-th Balkan Mathematical Olympiad

Belgrade, Yugoslavia – May 5, 2001

1. Let  $n$  be a positive integer. Prove that if  $a, b$  are integers greater than 1 such that  $ab = 2^n - 1$ , then the number  $ab - (a - b) - 1$  is of the form  $k \cdot 2^{2m}$ , where  $k$  is odd and  $m$  a positive integer.
2. Prove that a convex pentagon that satisfies the following two conditions must be regular:
  - (i) All its interior angles are equal;
  - (ii) The lengths of all its sides are rational numbers.
3. Let  $a, b, c$  be positive real numbers such that  $a + b + c \geq abc$ . Prove that

$$a^2 + b^2 + c^2 \geq abc\sqrt{3}.$$

4. A cube of edge 3 is divided into 27 unit cube cells. One of these cells is empty, while in the other cells there are unit cubes which are arbitrarily denoted by  $1, 2, \dots, 26$ . A *legal* move consists of moving a unit cube into a neighboring empty cell (two cells are neighboring if they share a face). Does there exist a finite sequence of legal moves after which any two cubes denoted by  $k$  and  $27 - k$  ( $k = 1, 2, \dots, 13$ ) will exchange their positions?