19-th Balkan Mathematical Olympiad

Antalya, Turkey – April 27, 2002

1. Points $A_1, A_2, \ldots, A_n$ $(n \geq 4)$, no three of which are collinear, are given on the plane. Some pairs of distinct points among them are connected by segments such that every point is connected to at least three other points. Prove that there exist an integer $k > 1$ and distinct points $X_1, X_2, \ldots, X_{2k}$ from the set $\{A_1, \ldots, A_n\}$ such that X_i is connected to X_{i+1} for $i = 1, 2, \ldots, 2k$, where $X_{2k+1} \equiv X_1$.

2. The sequence (a_n) is defined by $a_1 = 20$, $a_2 = 30$ and $a_{n+2} = 3a_{n+1} - a_n$ for every $n \geq 1$. Find all positive integers n for which $1 + 5a_n a_{n+1}$ is a perfect square.

3. Two circles with different radii intersect at A and B. Their common tangents MN and ST touch the first circle at M and S and the second circle at N and T. Show that the orthocenters of triangles AMN, AST, BMN, and BST are the vertices of a rectangle.

4. Determine all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all positive integers n

$$2n + 2001 \leq f(f(n)) + f(n) \leq 2n + 2002.$$