Yugoslav Team Selection Tests 2002

Selection Test for Balkan MO

Bečići, April 21

1. A man standing at the point $(1, 1)$ in the coordinate plane wants to find an object that lies at some point (α, β), where $\alpha \in \{1, 2, \dots, m\}$ and $\beta \in \{1, 2, \dots, n\}$. After finding the object, he will return to the starting point. Find the minimum time that is always sufficient for doing this job in terms of m and n. We assume that the man does not know at which point the object lies, and that he can move in any direction with velocity not exceeding 1.

2. Let p be the semiperimeter of a triangle ABC. Points E and F are taken on line AB such that $CE = CF = p$. Show that the circumcircle of the triangle EFC is tangent to the excircle of the triangle ABC that meets the side AB.

3. The sequence $(x_n)_{n=2}^{\infty}$ is defined by $x_2 = x_3 = 1$ and for $n \geq 3$,

$$(n+1)(n-2)x_{n+1} = n(n^2 - n - 1)x_n - (n-1)^3 x_{n-1}.$$

Prove that x_n is an integer if and only if n is a prime number.

Selection Test for IMO

Belgrade, June ?

1. Let a, b, c be nonnegative numbers such that $a^2 + b^2 + c^2 + abc = 4$. What is the maximum value of $a + b + c + abc$?

2. Let $ABCD$ be a convex quadrilateral with $\angle DAB = \angle ABC = \angle BCD$. Let O and H be the circumcenter and the orthocenter of triangle ABC, respectively. Prove that the points O, H, and D are collinear.

3. For any positive integer n, we denote by $f(n)$ the number of possible choices for plus and minus signs in the expression $1 \pm 2 \pm \cdots \pm n$ such that the value of the expression is 0.

   (a) Prove that $f(n) = 0$ for $n \equiv 1, 2 \pmod 4$.

   (b) Prove that $f(n) \geq 2^{\frac{n}{2}-1}$ for $n \equiv 0, 3 \pmod 4$.