

43-rd International Mathematical Olympiad

Glasgow, United Kingdom, July 19–30, 2002

First Day – July 24

1. Let n be a positive integer. Each point (x, y) in the plane, where x and y are nonnegative integers with $x + y = n$, is colored red or blue, subject to the following condition: If a point (x, y) is red, then so are all points (x', y') with $x' \leq x$ and $y' \leq y$. Let A be the number of ways to choose n blue points with distinct x -coordinates, and let B be the number of ways to choose n blue points with distinct y -coordinates. Prove that $A = B$.
(Colombia)
2. The circle S has center O , and BC is a diameter of S . Let A be a point of S such that $\angle AOB < 120^\circ$. Let D be the midpoint of the arc AB that does not contain C . The line through O parallel to DA meets the line AC at I . The perpendicular bisector of OA meets S at E and at F . Prove that I is the incenter of the triangle CEF .
(South Korea)
3. Find all pairs of positive integers $m, n \geq 3$ for which there exist infinitely many positive integers a such that

$$\frac{a^m + a - 1}{a^n + a^2 - 1}$$

is itself an integer.

(Romania)

Second Day – July 25

4. Let $n \geq 2$ be a positive integer, with divisors $1 = d_1 < d_2 < \dots < d_k = n$. Prove that $d_1 d_2 + d_2 d_3 + \dots + d_{k-1} d_k$ is always less than n^2 , and determine when it is a divisor of n^2 .
(Romania)
5. Find all functions f from the reals to the reals such that

$$(f(x) + f(z))(f(y) + f(t)) = f(xy - zt) + f(xt + yz)$$

for all real x, y, z, t .

(India)

6. Let $n \geq 3$ be a positive integer. Let $C_1, C_2, C_3, \dots, C_n$ be unit circles in the plane, with centers $O_1, O_2, O_3, \dots, O_n$ respectively. If no line meets more than two of the circles, prove that

$$\sum_{1 \leq i < j \leq n} \frac{1}{O_i O_j} \leq \frac{(n-1)\pi}{4}. \quad (\text{Ukraine})$$