20-th Balkan Mathematical Olympiad

Tirana, Albania – May 4, 2003

1. Does there exist a set B of 4004 distinct natural numbers, such that for any subset A of B containing 2003 elements, the sum of the elements of A is not divisible by 2003? *(FYR Macedonia)*

2. Let ABC be a triangle with $AB \neq AC$. The tangent at A to the circumcircle of the triangle ABC meets the line BC at D. Let E and F be the points on the perpendicular bisectors of the segments AB and AC respectively, such that BE and CF are both perpendicular to BC. Prove that the points D, E, and F are collinear. *(Romania)*

3. Find all functions $f : \mathbb{Q} \to \mathbb{R}$ which satisfy the following conditions:

   (i) $f(x+y) - yf(x) - xf(y) = f(x)f(y) - x - y + xy$ for all $x, y \in \mathbb{Q}$;

   (ii) $f(x) = 2f(x+1) + 2 + X$ for all $x \in \mathbb{Q}$;

   (iii) $f(1) + 1 > 0$. *(Cyprus)*

4. Let m and n be coprime odd positive integers. A rectangle $ABCD$ with $AB = m$ and $AD = n$ is divided into mn unit squares. Let $A_1, A_2, \ldots, A_k$ be the consecutive points of intersection of the diagonal AC with the sides of the unit squares (where $A_1 = A$ and $A_k = C$). Prove that

$$\sum_{j=1}^{k-1} (-1)^{j+1} A_j A_{j+1} = \frac{\sqrt{m^2+n^2}}{mn} .$$

(Bulgaria)