

# 44-th International Mathematical Olympiad

Tokyo, Japan – July 11–19, 2003

*First Day – July 13*

1. Let  $A$  be 101-element subset of the set  $S = \{1, 2, \dots, 1000000\}$ . Prove that there exist numbers  $t_1, t_2, \dots, t_{100}$  in  $S$  such that the sets

$$A_j = \{x + t_j | x \in A\}, \quad j = 1, 2, \dots, 100$$

are pairwise disjoint. *(Brazil)*

2. Determine all pairs  $(a, b)$  of positive integers such that

$$\frac{a^2}{2ab^2 - b^3 + 1}$$

is a positive integer. *(Bulgaria)*

3. Each pair of opposite sides of a convex hexagon has the following property: the distance between their midpoints is equal to  $\sqrt{3}/2$  times the sum of their lengths.

Prove that all the angles of the hexagon are equal. *(Poland)*

*Second Day – July 14*

4. Let  $ABCD$  be a cyclic quadrilateral. Let  $P, Q, R$  be the feet of the perpendiculars from  $D$  to the lines  $BC, CA, AB$ , respectively. Show that  $PQ = QR$  if and only if the bisectors of  $\angle ABC$  and  $\angle ADC$  are concurrent with  $AC$ . *(Finland)*

5. Let  $n$  be a positive integer and let  $x_1 \leq x_2 \leq \dots \leq x_n$  be real numbers

(a) Prove that

$$\left( \sum_{i,j=1}^n |x_i - x_j| \right)^2 \leq \frac{2(n^2 - 1)}{3} \sum_{i,j=1}^n (x_i - x_j)^2.$$

(b) Show that the equality holds if and only if  $x_1, \dots, x_n$  is an arithmetic sequence. *(Ireland)*

6. Let  $p$  be a prime number. Prove that there exists a prime number  $q$  such that for every integer  $n$ , the number  $n^p - p$  is not divisible by  $q$ . *(France)*