43-rd Federal Mathematical Competition of Serbia and Montenegro 2003

High School

Novi Sad, April 19, 2003

Time allowed 4 hours.
Each problem is worth 25 points.

1-st Grade

1. Find the number of solutions to the equation

$$ {x_1}^4 + {x_2}^4 + \ldots + {x_{10}}^4 = 2011 $$

   in the set of positive integers.

2. Let AB be a segment of length 2003 in a coordinate plane. Determine the maximum possible number of unit squares with vertices in the lattice points whose intersection with the segment AB is non-empty.

3. Let a,b and c be the side lengths of a triangle with angles $\alpha = 40^\circ$, $\beta = 60^\circ$ and $\gamma = 80^\circ$. Prove that $a(a+b+c) = b(b+c)$.

4. An acute angle with the vertex O and the rays Op_1 and Op_2 is given in a plane. Let k_1 be a circle with center on Op_1 which is tangent to Op_2, and let k_2 be the circle which is tangent to Op_1 and Op_2 and externally to circle k_1. Find the locus of points of tangency between k_1 and k_2 when the center of k_1 moves along the ray Op_1.

2-nd Grade

1. Let ABC be a triangle with sides a, b and c and the area S.

   (a) Prove that there is a triangle $A_1B_1C_1$ with sides $\sqrt{a}$, $\sqrt{b}$ and $\sqrt{c}$.

   (b) If S_1 is the area of $\triangle A_1B_1C_1$, prove that ${S_1}^2 \geq \frac{S\sqrt{3}}{4}$.

2. Let $ABCD$ be a square inscribed in a circle k and P be an arbitrary point of k. Prove that at least one of segments PA, PB, PC and PD has irrational length.

3. Let $ABCD$ be a rectangle. Find the set of all points P between the parallel lines AB and CD such that $\angle APB = \angle CPD$.

4. A subset S of $\mathbb{N}$ has the following properties:

   (i) Among any 2003 consecutive natural numbers, at least one is in S;

(ii) If $n \in S$ and $n > 1$, then $[n/2] \in S$ as well.

Prove that $S = \mathbb{N}$.

3-rd and 4-th Grades

1. Prove that the number $\left[(5+\sqrt{35})^{2n-1}\right]$ is divisible by 10^n for each $n \in \mathbb{N}$.

2. Let $f : [0,1] \to \mathbb{R}$ be a function with the following properties:

   (i) $f(x) \geq 0$ for all $x \in [0,1]$;

   (ii) $f(1) = 1$,

   (iii) if $x_1, x_2 \in [0,1]$ and $x_1 + x_2 \leq 1$, then $f(x_1) + f(x_2) \leq f(x_1 + x_2)$.

   Prove that for all $x \in [0,1]$ it holds that $f(x) \leq 2x$.

3. A circle k and a point P outside the circle are given. Let s be an arbitrary line passing through P and intersecting k at points A and B. Let M and N be the midpoints of the two arcs determined by$A$ and B and let C be the point on segment AB such that $PC^2 = PA \cdot PB$. Prove that $\angle MCN$ does not depend on the choice of s.

4. Let n be an even number and S be the set of all arrays of 0 and 1 of length n. Prove that S can be partitioned into disjoint three-element subsets such that: for any three arrays $(a_i), (b_i), (c_i)$ in the same subset and all $i = 1, 2, \ldots, n$, the number $a_i + b_i + c_i$ is even.

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