44-th Federal Mathematical Competition of Serbia and Montenegro 2004

High School

Šabac, April 17, 2004

Time allowed 4 hours.
Each problem is worth 25 points.

1-st Grade

1. Find all pairs of positive integers (a, b) such that $5a^b - b = 2004$.

2. In a triangle ABC, points D and E are taken on rays CB and CA respectively so that $CD = CE = \dfrac{AC + BC}{2}$. Let H be the orthocenter of the triangle, and P be the midpoint of the arc AB of the circumcircle of ABC not containing C. Prove that the line DE bisects the segment HP.
(R. Stanojević)

3. If a, b, c are positive numbers such that $abc = 1$, prove the inequality
$$\frac{1}{\sqrt{b + \frac{1}{a} + \frac{1}{2}}} + \frac{1}{\sqrt{c + \frac{1}{b} + \frac{1}{2}}} + \frac{1}{\sqrt{a + \frac{1}{c} + \frac{1}{2}}} \geq \sqrt{2}.$$
(R. Stanojević)

4. A set S of 100 points, no four in a plane, is given in space. Prove that there are no more than $4 \cdot 101^2$ tetrahedra with the vertices in S, such that any two of them have at most two vertices in common.
(R. Doroslovački)

2-nd Grade

1. Suppose that a, b, c are positive numbers such that $\dfrac{a}{b} + \dfrac{b}{c} + \dfrac{c}{a}$ is an integer. Show that abc is a perfect cube.
(D. Đukić)

2. Let r be the inradius of an acute triangle. Prove that the sum of the distances from the orthocenter to the sides of the triangle does not exceed $3r$.
(R. Stanojević)

3. Let M, N, P be arbitrary points on the sides BC, CA, AB respectively of an acute-angled triangle ABC. Prove that at least one of the following inequalities is satisfied:
$$NP \geq \frac{1}{2}BC, \quad PM \geq \frac{1}{2}CA, \quad MN \geq \frac{1}{2}AB.$$
(D. Đukić)

4. Baron Minchausen talked to a mathematician. Baron said that in his country from any town one can reach any other town by a road. Also, if one makes a circular trip from any town, one passes through an odd number of other towns. By this, as an answer to the mathematician's question, baron said that each town is counted as many times as it is passed through. Baron also added that the same number of roads start at each town in his country, except for the town where he was born, at which a smaller number of roads start. Then the mathematician said that baron lied. How did he conclude that? *(V. Baltić)*

3-rd and 4-th Grades

1. In a triangle ABC of the area S, point H is the orthocenter, D, E, F are the feet of the altitudes from A, B, C, and P, Q, R are the reflections of A, B, C in BC, CA, AB, respectively. The triangles DEF and PQR have the same area T. Given that $T > \frac{3}{5}S$, prove that $T = S$. *(R. Stanojević)*

2. The sequence (a_n) is determined by $a_1 = 0$ and

$$(n+1)^3 a_{n+1} = 2n^2(2n+1)a_n + 2(3n+1) \quad \text{for } n \geq 1.$$

Prove that infinitely many terms of the sequence are positive integers.
(Đ. Krtinić)

3. Let $A = \{1, 2, 3, \dots, 11\}$. How many subsets B of A are there, such that for each $n \in \{1, 2, \dots, 8\}$, if n and $n+2$ are in B then at least one of the numbers $n+1$ and $n+3$ is also in B? *(R. Doroslovački)*

4. The sequence (a_n) is given by $a_1 = x \in \mathbb{R}$ and $3a_{n+1} = a_n + 1$ for $n \geqslant 1$. Set

$$A = \sum_{n=1}^{\infty} \left[a_n - \frac{1}{6} \right], \quad B = \sum_{n=1}^{\infty} \left[a_n + \frac{1}{6} \right].$$

Compute the sum $A + B$ in terms of x. *(Đ. Krtinić)*