The First Danube Mathematical Cup

Calaraşi, Romania – December 10, 2005

1. Prove that the equation

$$4x^3 - 3x + 1 = 2y^2$$

   has at least 31 solutions in positive integers x and y with $x \le 2005$.

2. Prove that the sum

$$S_n = \binom{n}{1} + \binom{n}{3} \cdot 2005 + \binom{n}{5} \cdot 2005^2 + \ldots = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2k+1} \cdot 2005^k$$

   is divisible by 2^{n-1} for every positive integer n.

3. From a point A outside a circle $\mathcal{C}$ with center O, tangents AS and AT to the circle are constructed, where $S, T \in \mathcal{C}$. A point M, different from S and T, is arbitrarily selected on $\mathcal{C}$. The line MA intersects the line through S perpendicular to MO in point P. Show that the point symmetric to S with respect to P lies on the line MT.

4. Consider a board with $2(2^n - 1)$ rows and k columns, where $k, n \in \mathbb{N}$. A coloring of the board with two colors is called *admissible* if for every two columns the following condition holds: The number of rows intersecting the two columns in two cells of the same color does not exceed $2^n - 1$.

   For every n, determine the maximum value of k for which an admissible coloring exists.

Time for work: $4\frac{1}{2}$ hours.