

45-th Federal Mathematical Competition of Serbia and  
Montenegro 2005

High School  
Budva, April 16, 2005

*Time allowed 4 hours.  
Each problem is worth 25 points.*

**1-st Grade**

1. Find all positive integers  $n$  with the following property: For every positive divisor  $d$  of  $n$ ,  $d + 1$  divides  $n + 1$ .
2. Let  $ABC$  be an acute triangle. Circle  $k$  with diameter  $AB$  intersects  $AC$  and  $BC$  again at  $M$  and  $N$  respectively. The tangents to  $k$  at  $M$  and  $N$  meet at point  $P$ . Given that  $CP = MN$ , determine  $\angle ACB$ .
3. If  $x, y, z$  are nonnegative numbers with  $x + y + z = 3$ , prove that

$$\sqrt{x} + \sqrt{y} + \sqrt{z} \geq xy + yz + zx.$$

4. There are  $c$  red,  $p$  blue, and  $b$  white balls on a table. Two players  $A$  and  $B$  play a game by alternately making moves. In every move, a player takes two or three balls from the table. Player  $A$  begins. A player wins if after his/her move at least one of the three colors no longer exists among the balls remaining on the table. For which values of  $c, p, b$  does player  $A$  have a winning strategy?

**2-nd Grade**

1. Let  $A$  and  $b$  be positive integers and  $K = \sqrt{\frac{a^2 + b^2}{2}}$ ,  $A = \frac{a + b}{2}$ . If  $\frac{K}{A}$  is a positive integer, prove that  $a = b$ .
2. Every square of a  $3 \times 3$  board is assigned a sign  $+$  or  $-$ . In every move, one square is selected and the signs are changed in the selected square and all the neighboring squares (two squares are neighboring if they have a common side). Is it true that, no matter how the signs were initially distributed, one can obtain a table in which all signs are  $-$  after finitely many moves?
3. In a triangle  $ABC$ ,  $D$  is the orthogonal projection of the incenter  $I$  onto  $BC$ . Line  $DI$  meets the incircle again at  $E$ . Line  $AE$  intersects side  $BC$  at point  $F$ . Suppose that the segment  $IO$  is parallel to  $BC$ , where  $O$  is the circumcenter of  $\triangle ABC$ . If  $R$  is the circumcenter and  $r$  the incenter of the triangle, prove that  $EF = 2(R - 2r)$ .

4. Inside a circle  $k$  of radius  $R$  some round spots are made. The area of each spot is 1. Every radius of circle  $k$ , as well as every circle concentric with  $k$ , meets no more than one spot. Prove that the total area of all the spots is less than

$$\pi\sqrt{R} + \frac{1}{2}R\sqrt{R}.$$

### 3-rd and 4-th Grades

1. If  $x, y, z$  are positive numbers, prove that

$$\frac{x}{\sqrt{y+z}} + \frac{y}{\sqrt{z+x}} + \frac{z}{\sqrt{x+y}} \geq \sqrt{\frac{3}{2}(x+y+z)}.$$

2. Suppose that in a convex hexagon, each of the three lines connecting the midpoints of two opposite sides divides the hexagon into two parts of equal area. Prove that these three lines intersect in a point.
3. Determine all polynomials  $p$  with real coefficients for which  $p(0) = 0$  and

$$f(f(n)) + n = 4f(n) \quad \text{for all } n \in \mathbb{N},$$

where  $f(n) = [p(n)]$ .

4. On each cell of a  $2005 \times 2005$  chessboard there is a marker. In each move, we are allowed to remove a marker which is a neighbor to an even number of markers (but at least one). Two markers are considered neighboring if their cells share a vertex.
- (a) Find the least number  $n$  of markers that we can end up with on the chessboard.
- (b) If we end up with this minimum number  $n$  of markers, prove that no two of them will be neighboring.