

## 23-rd Balkan Mathematical Olympiad

Agros, Cyprus – April 29, 2006

1. If  $a, b, c$  are positive numbers, prove the inequality

$$\frac{1}{a(1+b)} + \frac{1}{b(1+c)} + \frac{1}{c(1+a)} \geq \frac{3}{1+abc}.$$

2. A line  $m$  intersects the sides  $AB$ ,  $AC$  and the extension of  $BC$  beyond  $C$  of the triangle  $ABC$  at points  $D, F, E$ , respectively. The lines through points  $A, B, C$  which are parallel to  $m$  meet the circumcircle of triangle  $ABC$  again at points  $A_1, B_1, C_1$ , respectively. Show that the lines  $A_1E$ ,  $B_1F$ ,  $C_1D$  are concurrent.

3. Determine all triples  $(m, n, p)$  of positive rational numbers such that the numbers

$$m + \frac{1}{np}, \quad n + \frac{1}{pm}, \quad p + \frac{1}{mn}$$

are integers.

4. Given a positive integer  $m$ , consider the sequence  $(a_n)$  of positive integers defined by the initial term  $a_0 = a$  and the recurrent relation

$$a_{n+1} = \begin{cases} \frac{a_n}{2} & \text{if } a_n \text{ is even,} \\ a_n + m & \text{if } a_n \text{ is odd.} \end{cases}$$

Find all values of  $a$  for which this sequence is periodic (i.e. there exists  $d > 0$  such that  $a_{n+d} = a_n$  for all  $n$ ).