47-th International Mathematical Olympiad

Ljubljana, Slovenia, July 6–18, 2006

First Day – July 12

1. Let ABC be a triangle with incenter I. A point P in the interior of the triangle satisfies

$$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$

Show that $AP \geq AI$, and that equality holds if and only if $P = I$.
(South Korea)

2. Let $\mathcal{P}$ be a regular 2006-gon. A diagonal of $\mathcal{P}$ is called *good* if its endpoints divide the boundary of $\mathcal{P}$ into two parts, each composed of an odd number of sides of $\mathcal{P}$. The sides of $\mathcal{P}$ are also called good.

Suppose $\mathcal{P}$ has been dissected into triangles by 2003 diagonals, no two of which have a common point in the interior of $\mathcal{P}$. Find the maximum number of isosceles triangles having two good sides that could appear in such a configuration. *(Serbia)*

3. Determine the least real number M such that the inequality

$$\left| ab(a^2 - b^2) + bc(b^2 - c^2) + ca(c^2 - a^2) \right| \leq M(a^2 + b^2 + c^2)^2$$

holds for all real numbers a, b and c. *(Ireland)*

Second Day – July 13

4. Determine all pairs (x, y) of integers such that

$$1 + 2^x + 2^{2x+1} = y^2.$$

(United States of America)

5. Let $P(x)$ be a polynomial of degree $n > 1$ with integer coefficients and let k be a positive integer. Consider the polynomial $Q(x) = P(P(\ldots P(P(x)) \ldots))$, where P occurs k times. Prove that there are at most n integers t such that $Q(t) = t$. *(Romania)*

6. Assign to each side b of a convex polygon $\mathcal{P}$ the maximum area of a triangle that has b as a side and is contained in $\mathcal{P}$. Show that the sum of the areas assigned to the sides of $\mathcal{P}$ is at least twice the area of $\mathcal{P}$. *(Serbia)*