

# 48-th International Mathematical Olympiad

Hanoi, Vietnam, July 19–31, 2007

*First Day – July 25*

1. Real numbers  $a_1, a_2, \dots, a_n$  are given. For each  $i$  ( $1 \leq i \leq n$ ) define

$$d_i = \max\{a_j \mid 1 \leq j \leq i\} - \min\{a_j \mid i \leq j \leq n\}$$

and let  $d = \max\{d_i \mid 1 \leq i \leq n\}$ .

- (a) Prove that, for any real numbers  $x_1 \leq x_2 \leq \dots \leq x_n$ ,

$$\max\{|x_i - a_i| \mid 1 \leq i \leq n\} \geq \frac{d}{2}. \quad (*)$$

- (b) Show that there are real numbers  $x_1 \leq x_2 \leq \dots \leq x_n$  such that equality holds in (\*). *(New Zealand)*

2. Consider five points  $A, B, C, D$  and  $E$  such that  $ABCD$  is a parallelogram and  $BCED$  is a cyclic quadrilateral. Let  $\ell$  be a line passing through  $A$ . Suppose that  $\ell$  intersects the interior of the segment  $DC$  at  $F$  and intersects line  $BC$  at  $G$ . Suppose also that  $EF = EG = EC$ . Prove that  $\ell$  is the bisector of angle  $DAB$ . *(Luxembourg)*

3. In a mathematical competition some competitors are friends. Friendship is always mutual. Call a group of competitors a *clique* if each two of them are friends. (In particular, any group of fewer than two competitors is a clique.) The number of members of a clique is called its *size*.

Given that, in this competition, the largest size of a clique is even, prove that the competitors can be arranged in two rooms such that the largest size of a clique contained in one room is the same as the largest size of a clique contained in the other room. *(Russia)*

*Second Day – July 26*

4. In triangle  $ABC$  the bisector of angle  $BCA$  intersects the circumcircle again at  $R$ , the perpendicular bisector of  $BC$  at  $P$ , and the perpendicular bisector of  $AC$  at  $Q$ . The midpoint of  $BC$  is  $K$  and the midpoint of  $AC$  is  $L$ . Prove that the triangles  $RPK$  and  $RQL$  have the same area. *(Czech Republic)*

5. Let  $a$  and  $b$  be positive integers. Show that if  $4ab - 1$  divides  $(4a^2 - 1)^2$ , then  $a = b$ . *(United Kingdom)*

6. Let  $n$  be a positive integer. Consider

$$S = \{(x, y, z) \mid x, y, z \in \{0, 1, \dots, n\}, x + y + z > 0\}$$

as a set of  $(n + 1)^3 - 1$  points in three-dimensional space. Determine the smallest possible number of planes, the union of which contains  $S$  but does not include  $(0, 0, 0)$ . *(Netherlands)*