# 25-th Balkan Mathematical Olympiad

## Ohrid, FYR Macedonia – May 6, 2008

**1.** An acute-angled scalene triangle $ABC$ with $AC > BC$ is given. Let $O$ be its circumcenter, $H$ its orthocenter, and $F$ the foot of the altitude from $C$. Let $P$ be the point (other than $A$) on the line $AB$ such that $AF = PF$, and $M$ be the midpoint of $AC$. We denote the intersection of $PH$ and $BC$ by $X$, the intersection of $OM$ and $FX$ by $Y$, and the intersection of $OF$ and $AC$ by $Z$. Prove that the points $F, M, Y$ and $Z$ are concyclic.

**2.** Does there exist a sequence $a_1, a_2, \ldots$ of positive numbers satisfying both of the following conditions:

(i) $\sum_{i=1}^{n} a_i \leq n^2$ for every positive integer $n$;

(ii) $\sum_{i=1}^{n} \frac{1}{a_i} \leq 2008$ for every positive integer $n$?

**3.** Let $n$ be a positive integer. The rectangle $ABCD$ with side lengths $90n + 1$ and $90n + 5$ is partitioned into unit squares with sides parallel to the sides of $ABCD$. Let $S$ be the set of all points which are vertices of these unit squares. Prove that the number of lines which pass through at least two points from $S$ is divisible by 4.

**4.** Let $c$ be a positive integer. The sequence $a_1, a_2, \ldots$ is defined by $a_1 = c$ and $a_{n+1} = a_n^2 + a_n + c^3$ for every positive integer $n$. Find all values of $c$ for which there exist some integers $k \geq 1$ and $m \geq 2$ such that $a_k^2 + c^3$ is the $m$-th power of some positive integer.

Time allowed: 4.5 hours.
Each problem is worth 10 points.