Romanian Master in Mathematics
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Problem 1. Let ABC be an equilateral triangle. P is a variable point internal to the triangle and its perpendicular distances to the sides are denoted by a^2, b^2 and c^2 for positive real numbers a, b and c. Find the locus of points P so that a, b and c can be the sides of a non-degenerate triangle.

[U.K.]

Solution. The required locus is the interior of the inscribed circle of triangle ABC.

To prove this, embed the equilateral triangle in the Cartezian space $Oxyz$, as the set in the plane $x + y + z = 1$ described by $x, y, z \geq 1$. Let the feet of the perpendiculars from P to BC and CA be D and E respectively, and let the feet of the perpendiculars from P to the planes OBC and OCA be Q and R respectively. Then triangles PQD and PRE are similar, so $PQ : PR = PD : PE$; i.e. $x : y = a^2 : b^2$, where (x, y, z) are coordinates of P. In the same way we get $y : z = b^2 : c^2$, so we have $(a^2 : b^2 : c^2) = (x : y : z)$.

Now if a, b and c are the sides of a triangle, the Heron's formula states that the square of the area of that triangle is

$$\frac{1}{16}(a + b + c)(-a + b + c)(a - b + c)(a + b - c).$$

So this quantity is positive. The reverse is also true.

Multiplying the expression out, this means that a, b and c are the sides of a triangle if and only if

$$2\sum b^2c^2 - \sum a^4 > 0.$$

Since a^2, b^2, c^2 are proportional to x, y, z, it follows that a, b and c are the sides of a triangle if and only if

$$2(x^2 + y^2 + z^2) < (x + y + z)^2 = 1.$$

So the required locus of points is the intersection of the solid sphere $x^2 + y^2 + z^2 < 1/2$ with the plane $x + y + z = 1$; that is the interior of the inscribed circle of the equilateral triangle.

Remark. Using a^2, b^2, c^2 as baricentric coordinates for P, in an equilateral triangle of circumradius 1, one can calculate the distance from P to the incenter I, reducing thus the problem to an algebraic one. In fact one can see the similarity to the above solution.

Problem 2. Prove that any bijective function $f : \mathbb{Z} \to \mathbb{Z}$ can be written as $f = u + v$ where $u, v : \mathbb{Z} \to \mathbb{Z}$ are bijective functions.

[Romania]

Solution. To find u, v such that $f = u + v$ it is enough to consider the case $f =$ identity on $\mathbb{Z}$. For that it suffices to write the above relation as $\mathrm{id}_{\mathbb{Z}} = u \circ f^{-1} + v \circ f^{-1}$. Consider the following well-ordering of the nonzero integers: $\mathbb{Z}^* = \{1, -1, 2, -2, \ldots, n, -n, \ldots\}$.

Build the following table

Step	A	#	B
1	1	$+1$	2
2	-1	-2	-3
3	-2	-3	-5
4	3	$+4$	7
$\vdots$	$\vdots$	$\vdots$	$\vdots$
k	a_k	$\mathrm{sign}(a_k) \cdot k$	$b_k = a_k + \#(k)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

The inductive rule in completing the table is as follows: at step 1 write 1, the first in the ordering of $\mathbb{Z}^*$, in column A, in column # put the number of the step, that is 1, with the sign from A, and in column B the sum from A and #. Suppose now that row of step i has been completed. Write on row $i+1$ in column A the first integer in the ordering of $\mathbb{Z}^*$ that has not yet been used in A nor B, in column # the number $i + 1$ with the sign given by that of the number just written in A, and in B the sum of A and #.

It is easy to see that in this manner we get an infinite array where $A \cup B = \mathbb{Z}^*$ and $A \cap B = \emptyset$, while elements in A and B do not repeat.

Define now $u(0) = v(0) = 0$ and for $x \in \mathbb{Z}$

• for $x = a_i \in A$ (meaning that x is in column A and row i), take $u(x) = -\#(i), v(x) = b_i$;

• for $x = b_j \in B$, take $u(x) = \#(j), v(x) = a_j$.

Obviously u and v are both bijections from $\mathbb{Z}$ to $\mathbb{Z}$ and $\mathrm{id}_{\mathbb{Z}} = u + v$. ■

Problem 3. Given positive integer $a > 1$, prove that any positive integer N has a multiple in the sequence

$$(a_n)_{n\geq 1}, \quad a_n = \left\lfloor \frac{a^n}{n} \right\rfloor.$$

[Romania]

Solution. In what follows, all literals will represent non-negative integers. The solution makes use of specific values for n, carefully chosen to facilitate the computation of the *floor* function.

Clearly, there exist $e \geq 0$, $q \geq 1$ and

$$M = a^{a^e - e}q, \quad \gcd(q, a) = 1,$$

such that M is a multiple of N.

Let us consider values $n = a^e p$, with p prime, $p > M$. Then, by Fermat's little Theorem ($p > M \geq a$, so $\gcd(a, p) = 1$)

$$a^{a^e(p-1)} - 1 = (a^{p-1})^{a^e} - 1 \equiv 0 \pmod{p}, \text{ so } a^n = a^{a^e}kp + a^{a^e},$$

therefore, as $n = a^e p > a^e M \geq a^{a^e}$

$$a_n = \left\lfloor \frac{a^n}{n} \right\rfloor = a^{a^e - e}k.$$

On the other hand, $kp = a^{a^e(p-1)} - 1$. Assuming $p - 1 = m\varphi(q)$ we have $a^{\varphi(q)} \equiv 1 \pmod{q}$[1] therefore $kp \equiv 0 \pmod{q}$, so q divides kp. But $p > M > q$, so $\gcd(q, p) = 1$, hence q divides k, so M (and *a fortiori* N) divides a_n.

We are left to prove that we can find such $p - 1 = m\varphi(q)$, that is, $p > M$ must belong to the arithmetic sequence of first-term 1 and ratio $\varphi(q)$. **The existence of such p is guaranteed by Dirichlet's Theorem**[2] and that should suffice in an international math competition. ■

Remarks. We will however, for self-containment, present a proof for this particular case of Dirichlet's Theorem [3]

An arithmetical sequence of first-term 1 and ratio r contains infinitely many primes (assume $r > 2$, as $r = 1$ or $r = 2$ makes it trivially true).

We will denote by d, $1 \leq d < r$, any (proper) divisor of r. Let us consider the polynomial $X^r - 1 \in \mathbb{Z}(X)$, factored in irreducible polynomials. Its roots (the *r-roots of unity*) are

$$\cos\frac{2k\pi}{r} + i\sin\frac{2k\pi}{r}, \text{ with } 1 \leq k \leq r,$$

[1] φ is the Euler *totient* function, and $\gcd(q, a) = 1$.

[2] Dirichlet's Theorem asserts the existence of infinitely many primes in an arithmetic sequence of co-prime first-term and ratio.

[3] This effort is a personal improvement on a proof by A. Rotkiewicz.

and, for $k = 1$, the main *primitive* r-root of unity ζ cannot be the root of any polynomial $X^d - 1$. Therefore ζ must be root of an irreducible factor $f(X)$ for $X^r - 1$, which cannot be a factor for any $X^d - 1$.[4] Now

$$f(X) \text{ divides } \frac{X^r - 1}{X^d - 1} \text{ for all } d, \text{ and } f(X) = \prod_{i=1}^{\deg f} (X - z_i),$$

with z_i among the r-roots of unity, so $|z_i| = 1$. Therefore, for any $n > 2$

$$|f(n)| = \prod_{i=1}^{\deg f} |n - z_i| \geq \prod_{i=1}^{\deg f} |n - |z_i|| = (n-1)^{\deg f} > 1.$$

Assume now there are only finitely many such primes q, and take $n = r \prod q$.[5] As $|f(n)| > 1$, there exists p prime, dividing $f(n)$, and therefore dividing $\frac{n^r - 1}{n^d - 1}$ for all d. We then cannot have p dividing $n^d - 1$ for any d, because

$$X^{\frac{r}{d}} - 1 = (X-1)P(X), \; P(X) = (X-1)Q(X) + R, \; R = P(1) = \frac{r}{d},$$

so $\frac{n^r - 1}{n^d - 1} = P(n^d) = (n^d - 1)Q(n^d) + \frac{r}{d}$, while clearly $n^d - 1$ and $\frac{r}{d}$ are co-prime (as r divides n), therefore p cannot divide $\frac{r}{d}$.

This shows that $n^r \equiv 1 \pmod{p}$ and $n^d \not\equiv 1 \pmod{p}$ for any d, so $r = \mathrm{ord}_p(n)$. But $n^{p-1} \equiv 1 \pmod{p}$ (by Fermat's little Theorem), so we must have r dividing $p - 1$, that is, p belongs to the stated arithmetical sequence. However, $p \neq q$ for any q considered in the above, as $\gcd(p, n) = 1$, and thus we have found yet another such prime, contradiction. $\square$

[4]In fact (not needed here), all primitive roots, for $\gcd(k, r) = 1$, are the roots of a **same** irreducible factor $\Phi_r(X)$, of degree $\varphi(r)$, which is the *cyclotomic polynomial* of order r. Then $X^r - 1 = \prod_{d|r} \Phi_d(X)$, the product of the (irreducible) cyclotomic polynomials.

[5]By definition $\prod q := 1$ if no such primes were to be selected.

Problem 4. Prove that from among any $(n+1)^2$ points inside a square of sidelength positive integer n, one can pick three, such that the triangle determined by them has area no more than $\frac{1}{2}$.

[Romania]

Solution. Although the topic of the problem may somehow appear familiar, the solution involves a novel and ingenious mix of ideas, centered around estimating areas of triangles using simple convexity inequalities.

Denote by $A = n^2$ the area of the square, by $P = 4n$ the perimeter of the square, and by $N = (n+1)^2$ the number of points. The convex hull of the set of N points will be a convex k-gon (contained in the given square), $3 \leq k \leq N$, with $N - k$ points in its interior (if any three points are collinear, they will determine a triangle of area 0, thus rendering the result trivially).

We will make use of the following folklore result

Any triangulation of a (convex) k-gon, using m = N - k interior points, is made of t = (k - 2) + 2m = 2(N - 1) - k triangles.[6]

As the area of the convex hull k-gon is at most A, it follows, using an *averaging* argument, that there will exist a triangle Δ_f of area at most

$$\frac{A}{t} = \frac{A}{2(N-1)-k} = f(k).$$

On the other hand, as the perimeter of the convex hull k-gon is at most P, one can find a pair of consecutive sides, be them $\mathbf{a}$, $\mathbf{b}$, of lengths a, b, such that $\frac{a+b}{2} \leq \frac{P}{k}$ (this also is an *averaging* argument). Now, the area of the triangle Δ_g determined by $\mathbf{a}$, $\mathbf{b}$, is

$$\frac{1}{2} ab \sin\angle(\mathbf{a}, \mathbf{b}) \leq \frac{1}{2}\Big(\frac{a+b}{2}\Big)^2 \leq \frac{P^2}{2k^2} = g(k).$$

Clearly, the bounds for the areas of triangles Δ_f, Δ_g depend on k, but $f(k)$ is increasing, while $g(k)$ is decreasing, therefore the worst case occurs for the value calculated in k_0 where the graphs of f and g meet

$$\frac{A}{2(N-1)-k_0} = \frac{P^2}{2k_0^2}, \text{ so } k_0^2 = 16(n+1)^2 - 16 - 8k_0, \text{ hence } k_0 = 4n.$$

Both formulae f and g, calculated in k_0, yield the value $\frac{1}{2}$, as required. ■

Remarks. One can improve on the bound given by $g(k)$; in fact it may be proven that a triangle Δ_g of area at most $\frac{P^2}{2k^2}\sin\frac{2\pi}{k}$ can be found. However, the minimum value offered by $f(k)$ is greater than $\frac{1}{2}(\frac{n}{n+1})^2$, which converges

[6] The total sum of angles for the t triangles is $t\pi$; but the vertices contribute $(k-2)\pi$, while the interior points contribute $2m\pi$, therefore $t = (k-2) + 2m$.

to $\frac{1}{2}$ when n grows large, thus thwarting any attempt to improve on the $\frac{1}{2}$ bound. The issue is to improve on the bound given by $f(k)$, but it is difficult to find efficient ways to bound from above the size of a least-area triangle for small k.

The author is far from claiming the result is tight (for large n), although better estimates appear elusive; however the naïve attempt to use the pigeonhole principle in its simplest form (partition the side-n square into n^2 unit squares; then for any $2n^2 + 1$ points inside the square there will exist three within a unit square, thus determining a triangle of area at most $\frac{1}{2}$), necessitates almost twice as many points as those afforded in the problem (except for $n = 2$, when $2 \cdot 2^2 + 1 = (2 + 1)^2$). On the other hand, for $n = 1$, the result is best possible!

Moreover, using the $\frac{P^2}{2k^2} \sin \frac{2\pi}{k}$ bound for Δ_g, one can prove for $n = 2$ that there exists a triangle of area at most $\frac{4}{9}$ (the critical point k_0 is moving from value 8 to 7, when the correct answer is given by $f(7) = \frac{4}{9}$), a better bound than anything found in the literature!