

# SERBIAN MATHEMATICAL OLYMPIAD

for high school students

Novi Sad, 13.04.2009.

## First Day

1. In a scalene triangle  $ABC$ ,  $\alpha$  and  $\beta$  respectively denote the interior angles at vertices  $A$  and  $B$ . The bisectors of these two angles meet the opposite sides of the triangle at points  $D$  and  $E$ , respectively. Prove that the acute angle between the lines  $DE$  and  $AB$  does not exceed  $\frac{|\alpha-\beta|}{3}$ . *(Dušan Djukić)*
2. Find the smallest natural number which is a multiple of 2009 and whose sum of (decimal) digits equals 2009. *(Miloš Milosavljević)*
3. Determine the largest positive integer  $n$  for which there exist pairwise different sets  $S_1, S_2, \dots, S_n$  with the following properties:
  - 1°  $|S_i \cup S_j| \leq 2004$  for any two indices  $1 \leq i, j \leq n$ , and
  - 2°  $S_i \cup S_j \cup S_k = \{1, 2, \dots, 2008\}$  for any  $1 \leq i < j < k \leq n$ . *(Ivan Matić)*

Time allowed: 270 minutes.  
Each problem is worth 7 points.

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Novi Sad, 14.04.2009.

## Second Day

4. For any  $n \in \mathbb{N}$ , denote by  $A_n$  the set of permutations  $(a_1, a_2, \dots, a_n)$  of set  $\{1, 2, \dots, n\}$  satisfying

$$k \mid 2(a_1 + a_2 + \dots + a_k), \quad \text{for each } 1 \leq k \leq n.$$

Compute the number of elements of  $A_n$ . *(Vidan Govedarica)*

5. Let  $x, y, z$  be arbitrary positive numbers such that  $xy + yz + zx = x + y + z$ . Prove that

$$\frac{1}{x^2 + y + 1} + \frac{1}{y^2 + z + 1} + \frac{1}{z^2 + x + 1} \leq 1.$$

When does equality occur? *(Marko Radovanović)*

6. The incircle  $k$  of a scalene triangle  $ABC$  is centered at  $S$  and tangent to the sides  $BC, CA, AB$  in points  $P, Q, R$ , respectively. Lines  $QR$  and  $BC$  intersect at point  $M$ . A circle passing through  $B$  and  $C$  is tangent to  $k$  at point  $N$ . The circumcircle of triangle  $MNP$  meets the line  $AP$  at point  $L$  different from  $P$ . Prove that the points  $S, L$  and  $M$  are collinear. *(Djordje Baralić)*

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