The 27th Balkan Mathematical Olympiad

Chisinau, Republic of Moldova, May 4 2010

English version

PROBLEMS

Each problem is worth 10 points.

Time allowed is 4 hours 30 min.

Problem 1. Let a, b and c be positive real numbers. Prove that

$$\frac{a^2b(b-c)}{a+b}+\frac{b^2c(c-a)}{b+c}+\frac{c^2a(a-b)}{c+a}\geq 0$$

Problem 2. Let ABC be an acute triangle with orthocenter H, and let M be the midpoint of AC. The point C_1 on AB is such that CC_1 is an altitude of the triangle ABC. Let H_1 be the reflection of H in AB. The orthogonal projections of C_1 onto the lines AH_1, AC and BC are P, Q and R, respectively. Let M_1 be the point such that the circumcentre of triangle PQR is the midpoint of the segment MM_1.
Prove that M_1 lies on the segment BH_1.

Problem 3. A *strip* of width w is the set of all points which lie on, or between, two parallel lines distance w apart. Let S be a set of n $(n \geq 3)$ points on the plane such that any three different points of S can be covered by a strip of width 1.
Prove that S can be covered by a strip of width 2.

Problem 4. For each integer n $(n \geq 2)$, let $f(n)$ denote the sum of all positive integers that are at most n and not relatively prime to n.
Prove that $f(n + p) \neq f(n)$ for each such n and every prime p.