# $28^{\mathrm{TH}}$ BALKAN MATHEMATICAL OLYMPIAD

Iaşi, Romania – May 6, 2011

**1.** Let $ABCD$ be a cyclic quadrilateral which is not a trapezoid and whose diagonals meet at $E$. The midpoints of $AB$ and $CD$ are $F$ and $G$ respectively, and $\ell$ is the line through $G$ parallel to $AB$. The feet of the perpendiculars from $E$ onto the lines $\ell$ and $CD$ are $H$ and $K$, respectively. Prove that the lines $EF$ and $HK$ are perpendicular.

**2.** Given real numbers $x, y, z$ such that $x + y + z = 0$, show that

$$\frac{x(x+2)}{2x^2+1} + \frac{y(y+2)}{2y^2+1} + \frac{z(z+2)}{2z^2+1} \geq 0.$$

When does equality hold?

**3.** Let $S$ be a finite set of positive integers which has the following property: If $x$ is a member of $S$, then so are all positive divisors of $x$. A non-empty subset $T$ of $S$ is *good* if, whenever $x, y \in T$ and $x < y$, the ratio $y/x$ is a power of a prime number. A non-empty subset $T$ of $S$ is *bad* if, whenever $x, y \in T$ and $x < y$, the ratio $y/x$ is not a power of a prime number. A one-element set is considered both good and bad. Let $k$ be the largest possible size of a good subset of $S$. Prove that $k$ is also the smallest number of pairwise disjoint bad subsets whose union is $S$.

**4.** Let $ABCDEF$ be a convex hexagon of area 1, whose opposite sides are parallel. The lines $AB$, $CD$ and $EF$ meet in pairs to determine the vertices of a triangle. Similarly, the lines $BC$, $DE$ and $FA$ meet in pairs to determine the vertices of another triangle. Show that the area of at least one of these two triangles is not less than $\frac{3}{2}$.

Each problem is worth 10 points.
Time allowed: $4\frac{1}{2}$ hours.