Serbian Mathematical Olympiad 2011

for high school students

Belgrade, April 2–3, 2011

Problems and Solutions

Cover photo: Cathedral of Saint Sava, Belgrade

SHORT HISTORY AND SYSTEM

Mathematical competitions in Serbia have been held since 1958. In the first years only republic competitions within the former Yugoslavia, which Serbia was a part of, were held. The first Federal Mathematical Competition in Yugoslavia was held in Belgrade in 1960, and since then it was held regularly every year, skipping only 1999 for a non-mathematical reason. The system has undergone relatively few changes. The earliest Federal Competitions were organized for 3rd and 4th grades of high school only; 2nd grade was added in 1970, and 1st grade in 1974. Since 1982, 3rd and 4th grades compose a single category. After the breakdown of the old Yugoslavia in 1991, the entire system was continued in the newly formed FR Yugoslavia, later renamed Serbia and Montenegro. The separation of Montenegro finally made the federal competition senseless as such. Thus, starting with 2007, the federal competition and team selection exam are replaced by a two-day Serbian Mathematical Olympiad.

Today a mathematical competition season in Serbia consists of four rounds of increasing difficulty:

- Municipal round, held in early February. The contest consists of 5 problems, each 20 points worth, to be solved in 3 hours. Students who perform well qualify for the next round (50-60 points are usually enough).

- Regional round, held in late February in the same format as the municipal round. Each student's score is added to that from the municipal round. Although the number of students qualified for the state round is bounded by regional quotas, a total score of 110-120 should suffice.

- State (republic) round, held in late March in a selected town in the country. There are roughly 200 to 300 participants. The contest consists of 5 problems in 4 hours.

- Serbian Mathematical Olympiad (SMO), held in early April in a selected place in the country. The participants are selected through the state round: 26 from A category (distribution among grades: 3+5+8+10), 3 from B category (0+0+1+2), plus those members of the last year's olympic team who did not manage to qualify otherwise. Six most successful contestants are invited to the olympic team.

Since 1998, contests for each grade on the three preliminary rounds are divided into categories A (specialized schools and classes) and B (others). A student from category B is normally allowed to work the problems for category A instead. On the SMO, all participants work on the same problems.

The Serbian Mathematical Olympiad 2011 for high school students took place in Belgrade on April 2–3. There were 31 students from Serbia and 5 guest students from the specialized school "Kolmogorov" in Moscow. The average score on the contest was 9.75 points and all problems were fully solved by the contestants. Based on the results of the competition the team of Serbia for the 28-th Balkan Mathematical Olympiad and the 52-nd International Mathematical Olympiad was selected:

Teodor von Burg	Math High School, Belgrade	42 points
Filip Živanović	Math High School, Belgrade	23 points
Igor Spasojević	Math High School, Belgrade	18 points
Rade Špegar	Math High School, Belgrade	18 points
Stevan Gajović	Math High School, Belgrade	16 points
Stefan Mihajlović	HS "Svetozar Marković", Niš	16 points

In this booklet we present the problems and full solutions of the Serbian Mathematical Olympiad and the Balkan Mathematical Olympiad.

SERBIAN MATHEMATICAL OLYMPIAD

for high school students

Belgrade , 02.04.2011.

First Day

1. Let $n \geqslant 2$ be a natural number and suppose that positive numbers $a_0, a_1, \ldots, a_n$ satisfy the equality

$$(a_{k-1} + a_k)(a_k + a_{k+1}) = a_{k-1} - a_{k+1} \quad \text{for each } k = 1, 2, \ldots, n-1.$$

Prove that $a_n < \dfrac{1}{n-1}$. *(Dušan Djukić)*

2. Let n be an odd positive integer such that numbers $\varphi(n)$ and $\varphi(n+1)$ are both powers of two ($\varphi(n)$ denotes the number of natural numbers coprime to n and not exceeding n). Prove that $n+1$ is a power of two or $n = 5$. *(Marko Radovanović)*

3. Let H be the orthocenter and O be the circumcenter of an acute-angled triangle ABC. Points D and E are the feet of the altitudes from A and B, respectively. Lines OD and BE meet at point K, and lines OE and AD meet at point L. Let X be the second intersection point of the circumcircles of triangles HKD and HLE, and let M be the midpoint of side AB. Prove that points K, L and M are collinear if and only if X is the circumcenter of triangle EOD. *(Marko Djikić)*

Time allowed: 270 minutes.
Each problem is worth 7 points.

SERBIAN MATHEMATICAL OLYMPIAD

for high school students

Belgrade, 03.04.2011.

Second Day

4. Points M, X and Y are taken on sides AB, AC and BC respectively of a triangle ABC such that $AX = MX$ and $BY = MY$. Let K and L be the midpoints of segments AY and BX respectively, and let O be the circumcenter of triangle ABC. If points O_1 and O_2 are symmetric to point O with respect to K and L respectively, show that the points X, Y, O_1 and O_2 lie on a circle. *(Marko Djikić)*

5. Do there exist integers a, b and c greater than 2011 such that in the decimal system they satisfy

$$\left(a+\sqrt{b}\right)^c = \ldots 2010, 2011\ldots ?$$

(Miloš Milosavljević)

6. Set T consists of 66 points, and set P consists of 16 lines in the plane. We say that a point $A \in T$ and a line $l \in P$ form an *incident pair* if $A \in l$. Show that the number of incident pairs cannot exceed 159, and that there is such a configuration with exactly 159 incident pairs. *(Miloš Stojaković)*

Time allowed: 270 minutes.
Each problem is worth 7 points.

SOLUTIONS

1. The given equation is equivalent to

$$\frac{1}{a_k + a_{k+1}} = 1 + \frac{1}{a_{k-1} + a_k}$$

for each $k > 0$. It follows by induction that $\frac{1}{a_k+a_{k+1}} = k + \frac{1}{a_0+a_1}$ for $k > 0$, which implies $\frac{1}{a_{n-1}+a_n} > n - 1$ and therefore $a_n < \frac{1}{n-1}$.

2. If $n = \prod_{i=1}^{k} p_i^{r_k}$ is the factorization of n into primes, we have $\varphi(n) = \prod_{i=1}^{k} p_i^{r_k-1}(p_i - 1)$. Since $\varphi(n)$ has no odd prime divisors, we must have $a_i = 1$ and $p_i - 1 = 2^{b_i}$ for each i and some b_i. Number $2^{b_i} + 1$ can be prime only if b_i is a power of two, so $p_i = 2^{2^{c_i}} + 1$ for some distinct c_i.
Suppose that $n+1$ is not a power of two. Then from $\varphi(n+1)$ being a power of two we obtain that all odd prime divisors of $n+1$ are of the form $2^{2^{d_i}} + 1$. Hence

$$n = \prod_{i=1}^{k}(2^{2^{c_i}} + 1), \quad n + 1 = 2^t \prod_{j=1}^{l}(2^{2^{d_j}} + 1),$$

where all c_i and d_j are mutually distinct. We can assume without loss of generality that $c_1 < \cdots < c_k$ i $d_1 < \cdots < d_l$.
For any $m, M \in \mathbb{N}$, $m \leq M$, simple induction shows that

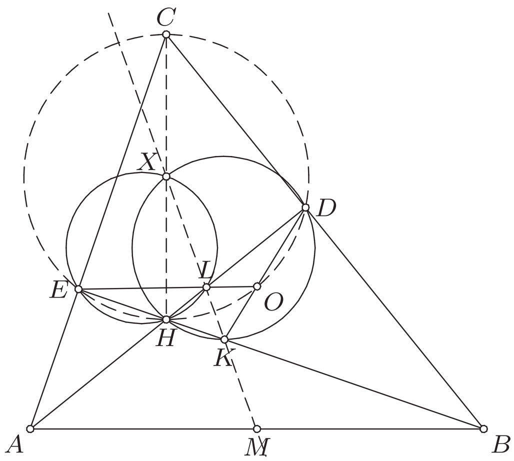
$$\frac{2^{2^m}+1}{2^{2^m}} < \prod_{i=m}^{M} \frac{2^{2^i}+1}{2^{2^i}} = \frac{2^{2^m}}{2^{2^m}-1} \cdot \frac{2^{2^{M+1}}-1}{2^{2^{M+1}}} < \frac{2^{2^m}}{2^{2^m}-1}.$$

This gives us

$$\frac{2^{2^{c_1}}+1}{2^{2^{c_1}}}2^c \leqslant n < \frac{2^{2^{c_1}}}{2^{2^{c_1}}-1}2^c \quad \text{i} \quad \frac{2^{2^{d_1}}+1}{2^{2^{d_1}}}2^d \leqslant n+1 < \frac{2^{2^{d_1}}}{2^{2^{d_1}}-1}2^d,$$

where $c = \sum_i 2^{c_i}$ and $d = t + \sum_j 2^{d_i}$. It follows that $c = d$. If $d_1 > c_1$, we have $\frac{2^{2^{d_1}}}{2^{2^{d_1}}-1} < \frac{2^{2^{c_1}}+1}{2^{2^{c_1}}}$, so $n + 1 < n$, a contradiction. Therefore $d_1 < c_1$ and thus $n + 1 \geqslant \frac{2^{2^{d_1}}+1}{2^{2^{d_1}}}2^c > \frac{2^{2^{c_1}}}{2^{2^{c_1}}-1}2^c > n$, so $\frac{n+1}{n} > \frac{2^{2^{d_1}}+1}{2^{2^{d_1}}} \cdot \frac{2^{2^{c_1}}-1}{2^{2^{c_1}}}$ and, since $n \geqslant 2^{2^{c_1}} + 1 \geqslant a^2 + 1$ for $2^{2^{d_1}} = a$, $\frac{n+1}{n} > \frac{(a+1)(a^2-1)}{a^3} = 1 + \frac{a^2-a-1}{a^3}$, from which we deduce $a^2 + 1 \leqslant n < \frac{a^3}{a^2-a-1}$. The only possibility is $a = 2$ and $n = 5$.

3. Suppose X is the circumcenter of $\triangle ODE$. Then $90^\circ - \angle KDE = 90^\circ - \angle ODE = \angle XEO = \angle XEL = \angle XHD = \angle XKD$ (all the angles are oriented), which means that $XK \perp DE$; Analogously, $XL \perp DE$, so K and L both lie on the perpendicular bisector of DE, hence $DEHO$ is an equilateral trapezoid and thus D, E, O, H lie on a circle.



On the other hand, if O lies on circle HDE, which is the circle with diameter CH, then the angles subtended by chords EH and OD are equal ($\angle ECH = \angle OCD$), so $DEHO$ is an equilateral trapezoid and hence $DL = EL$. Now $\angle EXH = \angle ELH = 2\angle EDH$ and analogously $\angle DXH = 2\angle DEH$, so X is the circumcenter of circle $DEOH$. Thus we have shown that X is the circumcenter of ODE if and only if D, E, O and H lie on a circle.

If points D, E, O, H are on a circle, then K, L and M belong to the perpendicular bisector of DE, which proves one direction of the problem. Now suppose that O lies outside the circle $CDHE$ (the case when O is inside the circle is similarly dealt with). Since $CO \perp DE$, we have $DL > LE$ and $EK > KD$, so K and L lie on different sides of the perpendicular bisector of DE, while M lies on the bisector. Therefore if K, L and M are collinear, M must lie between K and L. It follows that one of the points K, L is outside triangle ABC, whereas the other one is inside the triangle. However, when O is outside the quadrilateral $ABDE$, points K and L are both outside the triangle, otherwise they are both inside. This contradicts the assumption that M is on line KL, thus proving the other direction.

4. Consider the cartesian system with the origin at M and x-axis along the line AB. Let (a, b) and (c, d) be the coordinates of X and Y respectively. Since $AX = XM$ and $BY = YM$, the coordinates of A and B are $(2a, 0)$ and $(2c, 0)$, whereas those of K and L are $(a+\frac{c}{2}, \frac{d}{2})$ and $(c+\frac{a}{2}, \frac{b}{2})$, respectively. Point O has coordinates $(a+c, e)$ for some e, which gives us $O_1(a, d-e)$ and $O_2(c, b-e)$. Therefore, points O_1 and O_2 are symmetric to X and Y with respect to the line $y = \frac{b+d-e}{2}$. This means that X, Y and O_1, O_2 are vertices of a (possibly degenerate) equilateral trapezoid and hence lie on a circle.

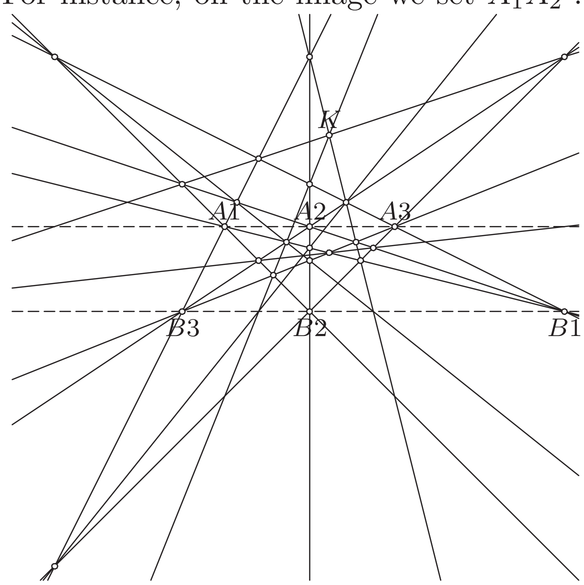
Remark. It is obvious from this solution that the statement of the problem remains valid when O is any point on the perpendicular bisector of AB.

5. We shall show that such numbers a, b and c exist. Note that the number $x = (a+\sqrt{b})^c + (a-\sqrt{b})^c$ is an integer. It is enough to choose a, b, c in such a way that x is a multiple of 10^4 and $7989.7989 > (a-\sqrt{b})^c > 7989.7988$.

For an odd c, number $x = 2a^c + 2\binom{c}{2}a^{c-2} + \cdots + 2\binom{c}{c-1}a$ is divisible by a, so taking any a that is divisible by 10^4 verifies the first condition. The second condition will be fulfilled if we take a and b so that $1 < a - \sqrt{b} < \sqrt{\frac{7989.7989}{7989.7988}}$ - for example, $a = 10^8$ and $b = (a-1)^2 - 1$. Indeed, let c be the smallest odd integer such that $(a - \sqrt{b})^c > 7989.7988$. This c is obviously greater than 2011 (in this case, $c = 1,797,184,159$) and $(a - \sqrt{b})^c < 7989.7989$.

6. Denote by $A_1, \ldots, A_{66}$ the points from T and by a_i the number of lines from P containing A_i. Then the number of pairs of lines intersecting at A_i equals $\binom{a_i}{2}$, and the number of incident pairs $I = \sum a_i$. Since any two lines meet in at most one point, we have $\sum_{i=1}^{66} \binom{a_i}{2} \leqslant \binom{16}{2} = 120$. Let b_k be the number of points from T which lie on exactly k lines from P. Then $\sum b_k = 66$, $\sum \binom{k}{2} b_k \leqslant 120$ and $I = \sum k b_k \leqslant \sum \frac{1}{2}\left(3 + \binom{k}{2}\right) b_k = \frac{1}{2}(3 \cdot 66 + 120) = 159$, because $3 + \binom{k}{2} \geq 2k$. Equality is attained when $b_k = 0$ for $k \notin \{2,3\}$, $b_2 = 39$ and $b_3 = 27$ - i.e. when the lines from P determine exactly 39 double and 27 triple intersection points.

An example of a configuration with 159 incident pairs can be constructed using Pappus' theorem. Take points A_1, A_2, A_3 on line a and B_1, B_2, B_3 on line $b \parallel a$, then draw 9 lines A_iB_j, $i, j \in \{1,2,3\}$. For instance, on the image we set $A_1A_2 : A_2A_3 : B_1B_2 : B_2B_3 = 2 : 2 : 3 : 6$, so among these lines no two are parallel and no three concurrent. By Pappus' theorem, the 9 lines determine 18 intersection points which are three-by-three collinear - so these determine another 6 lines. Together with these 6 lines, we have an image with 15 lines and 24 triple intersections. Moreover, three lines obtained by Pappus' theorem meet in a point (denoted by K), which gives us 25-th triple intersection. Draw one more line through two double intersection points only. The set P of the 16 drawn lines and set T consisting of the 27 triple intersection points and 39 remaining double intersection points determine 159 incident pairs.



The 28-th Balkan Mathematical Olympiad was held from May 4 to May 9 in Iasi in Romania. The results of Serbian contestants are given in the following table:

	1	2	3	4	Total	
Teodor von Burg	10	0	3	10	23	Silver Medal
Filip Živanović	10	1	0	2	13	Bronze Medal
Igor Spasojević	10	1	10	2	23	Silver Medal
Rade Špegar	4	9	4	10	27	Silver Medal
Stevan Gajović	10	2	3	0	15	Bronze Medal
Stefan Mihajlović	10	1	2	3	16	Bronze Medal

After the contest, 9 contestants (6 officially + 3 unofficially) with 30-40 points were awarded gold medals, 31 (16+15) with 17-29 points were awarded silver medals, and 46 (17+29) with 10-16 points were awarded bronze medals.

The unofficial ranking of the teams is given below:

Member Countries		Guest Teams	
1. Romania	170	Romania B	116
2. Turkey	148	Kazakhstan	113
3. Bulgaria	137	Italy	106
4. Serbia	117	United Kingdom	99
5. Greece	84	Tajikistan	78
6. Moldova	65	Turkmenistan	68
7. FYR Macedonia	64	France	66
8. Cyprus	27	Azerbaijan	64
9. Albania	25	Saudi Arabia	62
9. Montenegro	25	Indonesia	21

BALKAN MATHEMATICAL OLYMPIAD

Iaşi, Romania , 06.05.2011.

1. Let $ABCD$ be a cyclic quadrilateral which is not a trapezoid and whose diagonals meet at E. The midpoints of AB and CD are F and G respectively, and ℓ is the line through G parallel to AB. The feet of the perpendiculars from E onto the lines ℓ and CD are H and K, respectively. Prove that the lines EF and HK are perpendicular. *(United Kingdom)*

2. Given real numbers x, y, z such that $x + y + z = 0$, show that

$$\frac{x(x+2)}{2x^2+1} + \frac{y(y+2)}{2y^2+1} + \frac{z(z+2)}{2z^2+1} \geq 0.$$

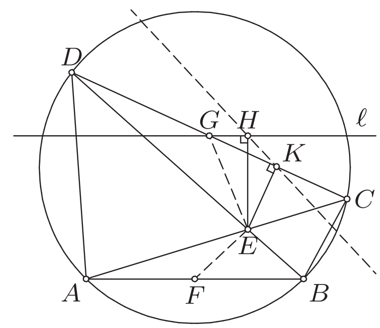
When does equality hold? *(Greece)*

3. Let S be a finite set of positive integers which has the following property: If x is a member of S, then so are all positive divisors of x. A non-empty subset T of S is *good* if, whenever $x, y \in T$ and $x < y$, the ratio y/x is a power of a prime number. A non-empty subset T of S is *bad* if, whenever $x, y \in T$ and $x < y$, the ratio y/x is not a power of a prime number. A one-element set is considered both good and bad. Let k be the largest possible size of a good subset of S. Prove that k is also the smallest number of pairwise disjoint bad subsets whose union is S. *(Bulgaria)*

4. Let $ABCDEF$ be a convex hexagon of area 1, whose opposite sides are parallel. The lines AB, CD and EF meet in pairs to determine the vertices of a triangle. Similarly, the lines BC, DE and FA meet in pairs to determine the vertices of another triangle. Show that the area of at least one of these two triangles is not less than $\frac{3}{2}$. *(Bulgaria)*

Time allowed: 270 minutes.
Each problem is worth 10 points.

SOLUTIONS



1. We may assume that $G-K-C$. Points E, G, H and K lie on the circle with diameter EG, so $\angle EHK = \angle EGK$. Since triangles EAB and EDC are similar ($\angle AEB = \angle DEC$ and $\angle EAB = \angle EDC$), so are triangles EFB and EGC. Thus $\angle EFB = \angle CGE = \angle KHE$, which together with $FB \perp HE$ yields $EF \perp KH$.

2. The desired inequality can be rewritten as

$$\frac{(2x+1)^2}{2x^2+1} + \frac{(2y+1)^2}{2y^2+1} + \frac{(2z+1)^2}{2z^2+1} \geq 3.$$

Assume without loss of generality that $|z| = \max\{|x|, |y|, |z|\}$. By the Cauchy-Schwarz inequality, we have

$$\frac{(2x+1)^2}{2x^2+1} + \frac{(2y+1)^2}{2y^2+1} \geq \frac{2(x+y+1)^2}{x^2+y^2+1} = \frac{2(1-z)^2}{x^2+y^2+1} \geq \frac{2(1-z)^2}{2z^2+1} = 3 - \frac{(2z+1)^2}{2z^2+1}$$

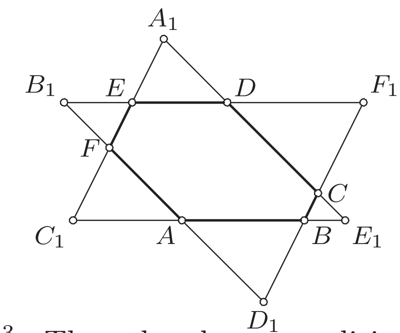
as we needed.

Equality holds when $x=y=z=0$ or $(x,y,z) = (-\frac{1}{2}, -\frac{1}{2}, 1)$ up to a permutation.

3. No two elements of a good set with k element can belong to a bad set, so we need at least k bad sets to cover S.

It remains to construct k bad sets that cover S. Let $p_1, \dots, p_n$ be all primes in S. Since S contains all divisors of its elements, each element of S must be of the form $x = p_1^{r_1} p_2^{r_2} \cdots p_n^{r_n}$, where $r_i \leq k-1$ for all i (because numbers x/p_i^j, $j = 0, \dots, r_i$, form a good subset of S with $r_i + 1$ elements).
For each such $x \in S$, define $\kappa(x) = r_1 + \dots + r_n$. If $x, y \in S$, $x < y$ belong to a good set, we have $1 \leq \kappa(y) - \kappa(x) \leq k-1$. Now consider sets $S_m = \{x \in S \mid \kappa(x) \equiv m \pmod{k}\}$, $m = 1, \dots, k$. It follows from above that each S_m is bad; Moreover, the S_m are pairwise disjoint and their union is S, so our construction is complete.

4. Let AB, CD and EF determine triangle $A_1C_1E_1$ and BC, DE, FA determine triangle $B_1D_1F_1$ ($CD \cap EF = \{A_1\}$, $DE \cap FA = \{B_1\}$, etc.). Denote $AB/F_1B_1 = a$, $BC/A_1C_1 = b$, $CD/B_1D_1 = c$, $DE/C_1E_1 = d$, $EF/D_1F_1 = e$, $FA/E_1A_1 = f$. Then $[ABD_1] = a^2[B_1D_1F_1]$ etc, so we obtain $[ABCDEF] = (1 - a^2 - c^2 - e^2)[B_1D_1F_1]$ and $[ABCDEF] = (1 - b^2 - d^2 - f^2)[A_1C_1E_1]$.



Ratios b, d, f can be expressed in terms of a, c, e: $b = \frac{BC}{A_1C_1} = \frac{D_1F_1 - D_1B - CF_1}{EF} \cdot \frac{EF}{A_1E+EF+FC_1} = \frac{1-a-c}{2-a-c-e}$ and analogously $d = \frac{1-c-e}{2-a-c-e}$ and $f = \frac{1-e-a}{2-a-c-e}$. If we denote $a + c + e = p$, we have $a^2 + c^2 + e^2 \geq \frac{1}{3}p^2$ and $b^2 + d^2 + f^2 = \frac{3-4p+p^2+a^2+c^2+e^2}{(2-p)^2} \geq \frac{1}{3}\left(\frac{3-2p}{2-p}\right)^2$.

Suppose that $[A_1C_1E_1] < \frac{3}{2}$ and $[B_1D_1F_1] < \frac{3}{2}$. Then the above equalities imply $a^2 + c^2 + e^2 < \frac{1}{3}$ and hence $p < 1$, but also $b^2 + d^2 + f^2 < \frac{1}{3}$ and hence $\frac{3-2p}{2-p} < 1$, a contradiction.