



Language: English

Saturday, April 28, 2012

Problem 1. Let A , B and C be points lying on a circle Γ with centre O . Assume that $\angle ABC > 90^\circ$. Let D be the point of intersection of the line AB with the line perpendicular to AC at C . Let ℓ be the line through D which is perpendicular to AO . Let E be the point of intersection of ℓ with the line AC , and let F be the point of intersection of Γ with ℓ that lies between D and E .

Prove that the circumcircles of triangles BFE and CFD are tangent at F .

Problem 2. Prove that

$$\sum_{\text{cyc}} (x+y)\sqrt{(z+x)(z+y)} \geq 4(xy+yz+zx),$$

for all positive real numbers x , y , and z .

The notation above means that the left-hand side is

$$(x+y)\sqrt{(z+x)(z+y)} + (y+z)\sqrt{(x+y)(x+z)} + (z+x)\sqrt{(y+z)(y+x)}.$$

Problem 3. Let n be a positive integer. Let $P_n = \{2^n, 2^{n-1} \cdot 3, 2^{n-2} \cdot 3^2, \dots, 3^n\}$. For each subset X of P_n , we write S_X for the sum of all elements of X , with the convention that $S_\emptyset = 0$ where \emptyset is the empty set. Suppose that y is a real number with $0 \leq y \leq 3^{n+1} - 2^{n+1}$.

Prove that there is a subset Y of P_n such that $0 \leq y - S_Y < 2^n$.

Problem 4. Let \mathbb{Z}^+ be the set of positive integers. Find all functions $f : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ such that the following conditions both hold:

- (i) $f(n!) = f(n)!$ for every positive integer n ,
- (ii) $m - n$ divides $f(m) - f(n)$ whenever m and n are different positive integers.

Each problem is worth 10 points.
Time allowed: 4 hours and 30 minutes.