Language: **English**

Day: **1**

EGMO | 2012
European Girls' Mathematical Olympiad

Thursday, April 12, 2012

Problem 1. Let ABC be a triangle with circumcentre O. The points D, E and F lie in the interiors of the sides BC, CA and AB respectively, such that DE is perpendicular to CO and DF is perpendicular to BO. (By *interior* we mean, for example, that the point D lies on the line BC and D is between B and C on that line.)

Let K be the circumcentre of triangle AFE. Prove that the lines DK and BC are perpendicular.

Problem 2. Let n be a positive integer. Find the greatest possible integer m, in terms of n, with the following property: a table with m rows and n columns can be filled with real numbers in such a manner that for any two different rows $[a_1, a_2, \ldots, a_n]$ and $[b_1, b_2, \ldots, b_n]$ the following holds:

$$\max(|a_1 - b_1|, |a_2 - b_2|, \ldots, |a_n - b_n|) = 1.$$

Problem 3. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\big(\, yf(x+y) + f(x) \,\big) = 4x + 2yf(x+y)$$

for all $x, y \in \mathbb{R}$.

Problem 4. A set A of integers is called *sum-full* if $A \subseteq A + A$, i.e. each element $a \in A$ is the sum of some pair of (not necessarily different) elements $b, c \in A$. A set A of integers is said to be *zero-sum-free* if 0 is the only integer that cannot be expressed as the sum of the elements of a finite nonempty subset of A.

Does there exist a sum-full zero-sum-free set of integers?

Time: 4 hours and 30 minutes
Each problem is worth 7 points

Language: **English**

Day: **2**

Friday, April 13, 2012

Problem 5. The numbers p and q are prime and satisfy

$$\frac{p}{p+1} + \frac{q+1}{q} = \frac{2n}{n+2}$$

for some positive integer n. Find all possible values of $q - p$.

Problem 6. There are infinitely many people registered on the social network *Mugbook*. Some pairs of (different) users are registered as *friends*, but each person has only finitely many friends. Every user has at least one friend. *(Friendship is symmetric; that is, if A is a friend of B, then B is a friend of A.)*

Each person is required to designate one of their friends as their *best friend*. If A designates B as her best friend, then (unfortunately) it does not follow that B necessarily designates A as her best friend. Someone designated as a best friend is called a 1*-best friend*. More generally, if $n > 1$ is a positive integer, then a user is an n*-best friend* provided that they have been designated the best friend of someone who is an $(n-1)$*-best friend*. Someone who is a k*-best friend* for every positive integer k is called *popular*.

(a) Prove that every popular person is the best friend of a popular person.

(b) Show that if people can have infinitely many friends, then it is possible that a popular person is not the best friend of a popular person.

Problem 7. Let ABC be an acute-angled triangle with circumcircle Γ and orthocentre H. Let K be a point of Γ on the other side of BC from A. Let L be the reflection of K in the line AB, and let M be the reflection of K in the line BC. Let E be the second point of intersection of Γ with the circumcircle of triangle BLM. Show that the lines KH, EM and BC are concurrent. *(The orthocentre of a triangle is the point on all three of its altitudes.)*

Problem 8. A *word* is a finite sequence of letters from some alphabet. A word is *repetitive* if it is a concatenation of at least two identical subwords (for example, *ababab* and *abcabc* are repetitive, but *ababa* and *aabb* are not). Prove that if a word has the property that swapping any two adjacent letters makes the word repetitive, then all its letters are identical. (Note that one may swap two adjacent identical letters, leaving a word unchanged.)