Language: English

Problem 1. Given a finite number of boys and girls, a *sociable set of boys* is a set of boys such that every girl knows at least one boy in that set; and a *sociable set of girls* is a set of girls such that every boy knows at least one girl in that set. Prove that the number of sociable sets of boys and the number of sociable sets of girls have the same parity. (Acquaintance is assumed to be mutual.)

Problem 2. Given a non-isosceles triangle ABC, let D, E, and F denote the midpoints of the sides BC, CA, and AB respectively. The circle BCF and the line BE meet again at P, and the circle ABE and the line AD meet again at Q. Finally, the lines DP and FQ meet at R. Prove that the centroid G of the triangle ABC lies on the circle PQR.

Problem 3. Each positive integer is coloured red or blue. A function f from the set of positive integers to itself has the following two properties:

(a) if $x \leq y$, then $f(x) \leq f(y)$; and

(b) if x, y and z are (not necessarily distinct) positive integers of the same colour and $x + y = z$, then $f(x) + f(y) = f(z)$.

Prove that there exists a positive number a such that $f(x) \leq ax$ for all positive integers x.

Each of the three problems is worth 7 points.
Time allowed $4\frac{1}{2}$ hours.

Language: English

Problem 4. Prove that there are infinitely many positive integers n such that $2^{2^n+1}+1$ is divisible by n but 2^n+1 is not.

Problem 5. Given a positive integer $n \geq 3$, colour each cell of an $n \times n$ square array with one of $\lfloor (n+2)^2/3 \rfloor$ colours, each colour being used at least once. Prove that there is some 1×3 or 3×1 rectangular subarray whose three cells are coloured with three different colours.

Problem 6. Let ABC be a triangle and let I and O denote its incentre and circumcentre respectively. Let ω_A be the circle through B and C which is tangent to the incircle of the triangle ABC; the circles ω_B and ω_C are defined similarly. The circles ω_B and ω_C meet at a point A' distinct from A; the points B' and C' are defined similarly. Prove that the lines AA', BB' and CC' are concurrent at a point on the line IO.

Each of the three problems is worth 7 points.
Time allowed $4\frac{1}{2}$ hours.