Serbian Mathematical Olympiad 2013

for high school students

Novi Sad, April 5–6, 2013

Problems and Solutions

Cover photo: Petrovaradin

The Serbian Mathematical Olympiad 2013 for high school students took place in Novi Sad on April 5–6. There were 31 students from Serbia and 5 guest students from Russia. Five members of the Serbian IMO team were selected through this contest, whereas the sixth was selected through an additional exam:

SRB1	Maksim Stokić	Math High School, Belgrade
SRB2	Žarko Randjelović	HS "Svetozar Marković", Niš
SRB3	Ivan Damnjanović	HS "Bora Stanković", Niš
SRB4	Bogdana Jelić	Math High School, Belgrade
SRB5	Lazar Radičević	Math High School, Belgrade
SRB6	Simon Stojković	Math High School, Belgrade

In this booklet we present the problems and full solutions of the Serbian Mathematical Olympiad with the Additional Team Selection Exam, and the Balkan Mathematical Olympiad.

Serbian MO 2013 – Problem Selection Committee

- Vladimir Baltić
- Bojan Bašić
- Dušan Djukić
- Miljan Knežević
- Miloš Milosavljević
- Marko Radovanović *(chairman)*
- Miloš Stojaković

Edited by Dušan Djukić

SERBIAN MATHEMATICAL OLYMPIAD

for high school students

Novi Sad , 05.04.2013.

First Day

1. Let k be a fixed natural number. A bijection $f:\mathbb{Z}\to\mathbb{Z}$ is such that if i and j are any integers satisfying $|i-j|\le k$, then $|f(i)-f(j)|\le k$. Prove that for any $i,j\in\mathbb{Z}$

$$|f(i)-f(j)| = |i-j|.$$

(Miljan Knežević)

2. Define

$$S_n = \left\{ \binom{n}{n}, \binom{2n}{n}, \binom{3n}{n}, \ldots, \binom{n^2}{n} \right\}, \quad \text{for } n\in\mathbb{N}.$$

a) Prove that there exist infinitely many composite natural numbers n such that S_n *is not* a complete set of residues modulo n.

b) Prove that there exist infinitely many composite natural numbers n such that S_n *is* a complete set of residues modulo n.

(Miloš Milosavljević)

3. Let M, N and P be the midpoints of sides BC, AC and AB respectively, and O be the circumcenter of an acute-angled triangle ABC. The circumcircles of triangles BOC and MNP intersect at distinct points X and Y inside the triangle ABC. Prove that

$$\measuredangle BAX = \measuredangle CAY.$$

(Marko Djikić)

Time allowed: 270 minutes.
Each problem is worth 7 points.

SERBIAN MATHEMATICAL OLYMPIAD

for high school students

Novi Sad, 06.04.2013.

Second Day

4. Find all $n \in \mathbb{N}$ for which it is possible to partition the set $\{1, 2, \ldots, 3n\}$ into n three-element subsets $\{a, b, c\}$ in which $b - a$ and $c - b$ are different numbers from the set $\{n - 1, n, n + 1\}$.

(Dušan Djukić)

5. Let A' and B' be the feet of the altitudes from A and B respectively of an acute-angled triangle ABC ($AC \neq BC$). Circle k through points A' and B' is tangent to side AB at point D. If the triangles ADA' and BDB' have equal areas, prove that

$$\measuredangle A'DB' = \measuredangle ACB.$$

(Miloš Milosavljević)

6. Determine the largest constant $K \in \mathbb{R}$ with the following property:
If $a_1, a_2, a_3, a_4 > 0$ are such that for any $i, j, k \in \mathbb{N}$, $1 \leq i < j < k \leq 4$ it holds that $a_i^2 + a_j^2 + a_k^2 \geq 2(a_i a_j + a_j a_k + a_k a_i)$, then

$$a_1^2 + a_2^2 + a_3^2 + a_4^2 \geq K(a_1 a_2 + a_1 a_3 + a_1 a_4 + a_2 a_3 + a_2 a_4 + a_3 a_4).$$

(Dušan Djukić)

Time allowed: 270 minutes.
Each problem is worth 7 points.

SOLUTIONS

1. The statement is trivial for $k = 1$, so we assume that $k \geq 2$. By an *interval* of length k we mean a set of the form $\{x, x+1, \ldots, x+k\}$, where $x \in \mathbb{Z}$. Two integers x and y are consecutive if and only if there exist intervals I_1 and I_2 of length k such that $I_1 \cap I_2 = \{x, y\}$. Then, since $f(I_1)$ and $f(I_2)$ also are intervals of length k and $\{f(x), f(y)\} = f(I_1) \cap f(I_2)$, we deduce that $f(x)$ and $f(y)$ are consecutive numbers. Therefore $|f(x+1) - f(x)| = 1$ for $x \in \mathbb{Z}$. Finally, taking injectivity into account, a simple induction on n shows that $|f(x+n) - f(x)| = n$.

2. (a) We show that $n = 2p$ satisfies the conditions, where p is an odd prime. We have

$$\binom{2kp}{2p} = k \prod_{i=1}^{p-1} \frac{2kp - i}{2p - i} \cdot (2k-1) \prod_{i=1}^{p-1} \frac{2kp - p - i}{p - i} \equiv k(2k-1) \pmod{p}.$$

In particular, $\binom{2kp}{2p}$ is divisible by p for $k \in \{\frac{p+1}{2}, p, 2p\}$, so S_{2p} has at least three elements divisible by p and cannot be a complete residue system.

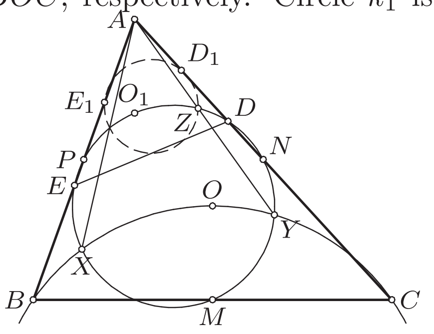
(b) We show that $n = p^2$ satisfies the conditions, where p is an odd prime. We have $\binom{kp^2}{p^2} = \prod_{i=0}^{p^2-1} \frac{kp^2-i}{p^2-i} = k \prod_{j=1}^{p-1} \frac{kp^2-jp}{jp} \cdot \prod_{p \nmid j} \frac{kp^2-i}{p^2-i}$, which yields modulo p^2

$$\binom{kn}{n} \equiv k \prod_{j=1}^{p-1} \frac{kp-j}{j} = k \prod_{j=1}^{p-1} \left(1 - \frac{kp}{j}\right) \equiv k - k^2 p \sum_{j=1}^{p-1} \frac{1}{j}.$$

Since $\sum_{j=1}^{p-1} \frac{1}{j} = \sum_{j=1}^{\frac{p-1}{2}} (\frac{1}{j} + \frac{1}{p-j}) = \sum_{j=1}^{\frac{p-1}{2}} \frac{p}{j(p-j)} \equiv 0 \pmod{p}$, it follows that $\binom{kp^2}{p^2} \equiv k \pmod{p^2}$.

Remark. There are other possibilities for numbers n: for instance, (a) $n = 8k + 6$ for $k \in \mathbb{N}$, and (b) $n = p^k$ for a prime p.

3. Denote by k_1 and k_2 the circles MNP and BOC, respectively. Circle k_1 is the nine-point circle of $\triangle ABC$ and passes through the feet D, E of the altitudes BD and CE and the midpoint O_1 of AH, where H is the orthocenter of $\triangle ABC$.



We claim that the second intersection point Z of AY and k_1 lies on the nine-point circle k_3 of triangle ADE. We assume that Z is between A and Y; the other case is analogous. Let D_1 and E_1 be the midpoints of AD and AE, respectively. Since $AY \cdot AZ = AD \cdot AN = AD_1 \cdot AC$, points Y, Z, C, D_1 lie on a circle, implying $\angle AZD_1 = \angle ACY$. Analogously, $\angle AZE_1 = \angle ABY$, and therefore $\angle D_1ZE_1 = \angle AZD_1 + \angle AZE_1 = \angle ACY + \angle ABY = \angle BYC - \angle BAC = \angle BAC$. Hence Z is on k_3.

Since O_1 is the circumcenter of $\triangle ADE$, the similarity mapping $\triangle ABC$ onto $\triangle ADE$ maps k_1 to k_2 and k_2 to k_3, so it takes point $X \in k_1 \cap k_2$ to point $Z \in k_2 \cap k_3$. Therefore $\angle BAX = \angle DAZ = \angle CAY$.

Second solution. Apply the inversion with center A and power $\frac{1}{2}AB \cdot AC$, and then reflect in the bisector of angle CAB. Points B, C and N, P go to N, P and B, C respectively. Furthermore, point O satisfies $\angle ANO = \angle APO = 90°$, so its image O' satisfies $\angle AO'B = \angle AO'C = 90°$, i.e. AO' is an altitude in $\triangle ABC$. It follows that circle $\omega_1(NO'P)$ maps to circle $\omega_2(BOC)$, and vice-versa. Therefore their intersection points X, Y map to each other, and consequently $\angle BAX = \angle CAY$.

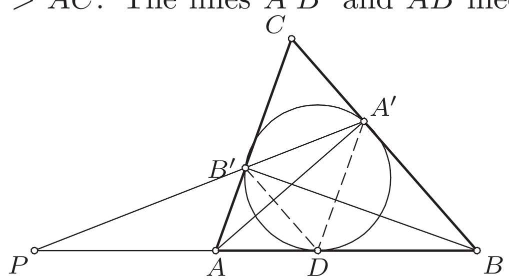
4. A required partition of the set $\{1, 2, \ldots, 3n\}$ corresponds to a partition of the vertices of a regular $3n$-gon $P_1P_2 \ldots P_{3n}$ into triples $\{A_i, B_i, C_i\}$ such that each of the triangles $A_iB_iC_i$ has the angles $\frac{n-1}{3n}\pi$, $\frac{n}{3n}\pi$ and $\frac{n+1}{3n}\pi$. There is a suitable numeration of the vertices of the $3n$-gon such that the vertices A_1, B_1, C_1 are precisely P_n, P_{2n-1}, P_{3n} in some order. In other words, there is no loss of generality in assuming that one of the triples in the partition of $\{1, 2, \ldots, 3n\}$ is the triple $\{n, 2n-1, 3n\}$.

One of the remaining $n-1$ triples must contain two numbers from the interval $[2n, 3n-1]$. These two numbers must be $2n$ and $3n-1$, so the only such triple not containing n is $\{n-1, 2n, 3n-1\}$.

Every other triple contains a number from each of the intervals $[1, n-2]$, $[n+1, 2n-2]$ and $[2n+1, 3n-2]$. The map $(a, b, c) \to (a, b-2, c-4)$ for $a < b < c$ now yields a required partition of the set $\{1, 2, \ldots, 3(n-2)\}$ into triples. Since such a partition is impossible for $n = 1$, a simple induction shows that it is impossible for any odd n.

On the other hand, for an even $n = 2m$ the triples $(2i-1, 2i+n, 2i+2n-1)$ and $(2i, 2i+n-1, 2i+2n)$ for $i = 1, \ldots, m$ fulfill the requirements.

5. Assume without loss of generality that $BC > AC$. The lines $A'B'$ and AB meet at a point P with A between P and B. The equality of the areas of ADA' and BDB' gives us $\frac{AD}{DB} = \frac{PB'}{PA'}$. It also holds that $PD^2 = PA' \cdot PB' = PA \cdot PB$, and hence $\frac{PD}{PB} = \frac{PA}{PD} = \frac{AD}{DB} = \frac{PB'}{PA'}$. The last equalities imply that $B'D \parallel BC$ and $A'D \parallel AC$, so $\angle A'DB' = \angle ACB$.



Second solution. Since $\angle CB'D = \alpha + \angle ADB' = \alpha + \angle B'A'D = \angle CA'D = x$, the Sine law in $\triangle A'B'D$ and $\triangle AB'D$ gives $BD = \frac{BA'\sin x}{\sin(\beta+x)}$ and $AD = \frac{AB'\sin x}{\sin(\alpha+x)}$, so $\frac{AD}{BD} = \frac{AB'\sin(\beta+x)}{BA'\sin(\alpha+x)} = \frac{\sin(\beta+x)\cos\alpha}{\sin(\alpha+x)\cos\beta}$. On the other hand, the condition $[ADA'] = [BDB']$ implies $\frac{AD}{DB} = \frac{d(B',AB)}{d(A',AB)} = \frac{\sin\alpha\cos\alpha}{\sin\beta\cos\beta}$, i.e. $\frac{\sin(\beta+x)}{\sin(\alpha+x)} = \frac{\sin\alpha}{\sin\beta}$. From here we easily obtain $\angle AB'D = x = \gamma$.

6. Let $\max\{a_1, a_2\} \leqslant a_3 \leqslant a_4$ and denote $a_2 = \beta^2$ and $a_3 = \gamma^2$, $\beta, \gamma \geqslant 0$. The problem condition implies $a_1 \leqslant (\gamma - \beta)^2$ and $a_4 \geqslant (\gamma + \beta)^2$.

Suppose for a moment that these two inequalities are in fact equalities. Then $a_1^2 + a_2^2 + a_3^2 + a_4^2 = 3(\beta^4 + 4\beta^2\gamma^2 + \gamma^4)$ and $a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_3a_4 = 3(\beta^4 + \beta^2\gamma^2 + \gamma^4)$. Furthermore, $\gamma \leqslant 2\beta$, hence $\frac{\beta^4+4\beta^2\gamma^2+\gamma^4}{\beta^4+\beta^2\gamma^2+\gamma^4} = 1 + \frac{3\beta^2\gamma^2}{\beta^4+\beta^2\gamma^2+\gamma^4} = 1 + \frac{3}{1+\frac{\beta^2}{\gamma^2}+\frac{\gamma^2}{\beta^2}} \geqslant \frac{11}{7}$, with equality for $\gamma = 2\beta$. Thus, in this case we have $K \geqslant \frac{11}{7}$, and equality is attained if $a_1 : a_2 : a_3 : a_4 = 1 : 1 : 4 : 9$.

It remains to show that we may indeed assume $a_1 = (\gamma - \beta)^2$ and $a_4 = (\gamma + \beta)^2$. Consider

$$F = a_1^2 + a_2^2 + a_3^2 + a_4^2 - \frac{11}{7}(a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_3a_4).$$

For fixed a_2, a_3, a_4, F is decreasing on a_1 for $a_1 < \frac{11}{14}(a_2 + a_3 + a_4)$, whereby $\frac{11}{14}(a_2 + a_3 + a_4) \geqslant \frac{11}{14}(\beta^2 + \gamma^2 + (\beta + \gamma)^2) \geqslant (\gamma - \beta)^2 \geqslant a_1$, so F will not increase if a_1 is changed to $(\gamma - \beta)^2$. Now we can assume without loss of generality that $a_1 \leqslant a_2$, i.e. $\beta \leqslant \gamma \leqslant 2\beta$. As above, for fixed a_1, a_2, a_3, F is increasing on a_4 for $a_4 > \frac{11}{14}(a_1 + a_2 + a_3)$, whereby $\frac{11}{14}(a_1 + a_2 + a_3) \leqslant \frac{11}{14}(\beta^2 + \gamma^2 + (\gamma - \beta)^2) \leqslant (\gamma + \beta)^2 \leqslant a_4$, so F will not increase if a_4 is changed to $(\gamma + \beta)^2$.

Additional IMO Team Selection Exam

Belgrade , 27.04.2013.

1. We call polynomials $A(x) = a_n x^n + \ldots + a_1 x + a_0$ and $B(x) = b_m x^m + \ldots + b_1 x + b_0$ $(a_n b_m \neq 0)$ *similar* if the following conditions hold:

(i) $n = m$;

(ii) There is a permutation π of the set $\{0, 1, \ldots, n\}$ such that $b_i = a_{\pi(i)}$ for each $i \in \{0, 1, \ldots, n\}$.

Let $P(x)$ and $Q(x)$ be similar polynomials with integer coefficients. Given that $P(16) = 3^{2012}$, find the smallest possible value of $|Q(3^{2012})|$.

(Miloš Milosavljević)

2. In an acute triangle ABC $(AB \neq AC)$ with angle α at the vertex A, point E is the nine-point center, and P a point on the segment AE. If $\angle ABP = \angle ACP = x$, prove that $x = 90^\circ - 2\alpha$.

(Dušan Djukić)

3. Let $p > 3$ be a given prime number. For a set $S \subseteq \mathbb{Z}$ and $a \in \mathbb{Z}$, define

$$S_a = \{x \in \{0, 1, ..., p-1\} \mid (\exists s \in S)\ x \equiv_p a \cdot s\}.$$

(a) How many sets $S \subseteq \{1, 2, ..., p-1\}$ are there for which the sequence $S_1, S_2, ..., S_{p-1}$ contains exactly two distinct terms?

(b) Determine all numbers $k \in \mathbb{N}$ for which there is a set $S \subseteq \{1, 2, ..., p-1\}$ such that the sequence $S_1, S_2, ..., S_{p-1}$ contains exactly k distinct terms.

(Milan Bašić, Miloš Milosavljević)

Time allowed: 270 minutes.
Each problem is worth 7 points.

SOLUTIONS

1. Since $3^{2012} \equiv 1 \pmod 5$, it holds that $Q(3^{2012}) \equiv Q(1) = P(1) \equiv P(16) \equiv 1 \pmod 5$, so $|Q(3^{2012})| \geqslant 1$.

Now we construct polynomials P and Q satisfying the conditions for which $Q(3^{2012}) = 1$. Set $P(x) = ax^2 + bx + c$ and $Q(x) = cx^2 + ax + b$. Denoting $m = 16$ and $n = 3^{2012}$, we want the system of equations

$$\begin{cases} am^2 + bm + c = n \\ cn^2 + an + b = 1 \end{cases}$$

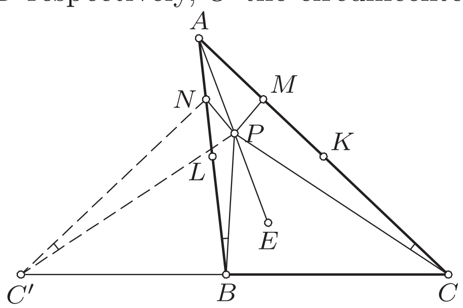
to have an integer solution (a, b, c). Substituting $c = n - am^2 - bm$ in the second equation yields $n^2(n-am^2-bm)+an+b = 1$, i.e. $n(m^2n-1)a+(mn^2-1)b = n^3-1$, so it suffices that this equation have an integer solution (a, b). The last condition is equivalent to $\gcd(n(m^2n - 1), mn^2 - 1) \mid n^3 - 1$, which is true: Indeed, if $d \mid n(m^2n - 1)$ and $d \mid mn^2 - 1$, then $d \mid n(m^2n - 1) - m(mn^2 - 1) = m - n$, and therefore $d \mid mn^2 - 1 + n^2(n - m) = n^3 - 1$.

Second solution. As in the first solution, $Q(3^{2012}) \neq 0$.

In order to achieve that $Q(3^{2012}) = 1$, knowing that $P(3^{2012}) \equiv Q(3^{2012}) \pmod{3^{2012} - 1}$, we impose the extra condition $P(3^{2012}) = P(16) = 3^{2012}$. The polynomial P is of the form $P(x) = (x - 16)(x - 3^{2012})S(x) + 3^{2012}$. Suppose that S is chosen in such a way that $P(x) = a_nx^n + \cdots + a_2x^2 + (c + 1)x + c$ for some $c \in \mathbb{Z}$; then taking $Q(x) = a_nx^n + \cdots + a_2x^2 + cx + (c + 1)$ would yield precisely $Q(3^{2012}) = P(3^{2012}) - (3^{2012} - 1) = 1$.

We find $S(x)$ in the form $ax + b$. The last two coefficients in $P(x)$ are $16 \cdot 3^{2012}a - (16 + 3^{2012})b$ and $16 \cdot 3^{2012}b + 3^{2012}$ respectively. Thus it suffices to take a and b so that $16 \cdot 3^{2012}a - (16 \cdot 3^{2012} + 16 + 3^{2012})b - 3^{2012} = 1$, which is possible because $16 \cdot 3^{2012}$ and $16 \cdot 3^{2012} + 16 + 3^{2012}$ are coprime.

2. Let K and L be the midpoints of AC and AB respectively, O the circumcenter of $\triangle ABC$, and $M \in AC$, $N \in AB$ such that $PM \parallel EK$ and $PN \parallel EL$. Then $\angle PMC = \angle EKC = \angle LKC - \angle LKE = 180° - \gamma - \angle CBO = 2\alpha + \beta - 90°$.



Consider the reflection C' of point C in the perpendicular bisector of MN ($C' \not\equiv B$ since $AB \neq AC$). Points B, P, N and C' lie on a circle for $\angle NC'P$

$= \angle NBP = x$. Thus $\beta - x = \angle CBP = \angle PNC' = \angle PMC = 2\alpha + \beta - 90^\circ$, and therefore the statement.

Second solution. The Ceva theorem in trigonometric form for point P in triangle ABC gives us $\frac{\sin\angle CAE}{\sin\angle EAB} = \frac{\sin\angle PBC}{\sin\angle PCB} = \frac{\sin(\beta - x)}{\sin(\gamma - x)}$. On the other hand, the same theorem for E in $\triangle AKL$ gives $\frac{\sin\angle CAE}{\sin\angle EAB} = \frac{\sin\angle AKE}{\sin\angle ELA} = \frac{\cos(\gamma-\alpha)}{\cos(\beta-\alpha)}$. These two equalities together imply

$$\begin{aligned} 0 &= \sin(\beta - x)\cos(\beta - \alpha) - \sin(\gamma - x)\cos(\gamma - \alpha) \\ &= \tfrac{1}{2}(\sin(2\beta - \alpha - x) + \sin(\alpha - x) - \sin(2\gamma - \alpha - x) - \sin(\alpha - x)) \\ &= \sin(\beta - \gamma)\cos(180^\circ - 2\alpha - x), \end{aligned}$$

so $x = 90^\circ - 2\alpha$.

3. Let g be a primitive root modulo p. The sequence $S_1, S_2, \dots, S_{p-1}$ is a permutation of $S_1, S_g, \dots, S_{g^{p-2}}$. Since the sequence $S_1, S_g, S_{g^2}, \dots$ is periodic with period $p-1$, its minimal period d divides $p - 1$ and the sets $S_1, S_g, \dots, S_{g^{d-1}}$ are mutually distinct. The multiplications in the sequel are modulo p.

(b) It follows from above that $k \mid p - 1$. On the other hand, for $k \mid p - 1$ we can take $S = \{1, g^k, g^{2k}, \dots\}$.

(a) The minimal period of the sequence $S_1, S_g, S_{g^2}, \dots$ is 2, so $S_0 = S_2 \neq S_1$. Therefore $x \in S \Rightarrow g^2x \in S$. If $a, ga \in S$, then all $g^n a$ $(n \in \mathbb{N}_0)$ are in S, i.e. $S = \{1, 2, \dots, p-1\}$ which is impossible. It follows that $x \in S \Rightarrow gx \notin S$, $g^2x \in S$. Thus S is either $\{1, g^2, g^4, \dots, g^{p-3}\}$ or $\{g, g^3, \dots, g^{p-2}\}$, so the answer is two.

BALKAN MATHEMATICAL OLYMPIAD

Agros, Cyprus , 30.06.2013.

1. In a triangle ABC, the excircle ω_a opposite A touches AB at P and AC at Q, and the excircle ω_b opposite B touches BA at M and BC at N. Let K be the projection of C onto MN, and let L be the projection of C onto PQ.
Show that the quadrilateral $MKLP$ is cyclic. *(Bulgaria)*

2. Determine all positive integers x, y and z such that

$$x^5 + 4^y = 2013^z.$$

(Serbia)

3. Let S be the set of positive real numbers. Find all functions $f : S^3 \to S$ such that, for all positive real numbers x, y, z and k, the following three conditions are satisfied:

(i) $xf(x, y, z) = zf(z, y, x)$;

(ii) $f(x, ky, k^2z) = kf(x, y, z)$;

(iii) $f(1, k, k + 1) = k + 1$. *(United Kingdom)*

4. In a mathematical competition some competitors are friends; friendship is always mutual, that is to say that when A is a friend of B, then also B is a friend of A. We say that $n \geq 3$ different competitors $A_1, A_2, \ldots, A_n$ form a *weakly-friendly cycle* if A_i is not a friend of A_{i+1} for $1 \leq i \leq n$ ($A_{n+1} = A_1$), and there are no other pairs of non-friends among the components of this cycle.
The following property is satisfied:

> for every competitor C, and every weakly-friendly cycle $\mathscr{S}$ of competitors not including C, the set of competitors D in $\mathscr{S}$ which are not friends of C has at most one element.

Prove that all competitors of this mathematical competition can be arranged into three rooms, such that every two competitors that are in the same room are friends. *(Serbia)*

Time allowed: 270 minutes.
Each problem is worth 10 points.

SOLUTIONS

1. We denote the angles of the triangle by α, β and γ as usual. Since $\angle KMP = 90^\circ - \frac{\beta}{2}$, it suffices to prove that $\angle KLP = 90^\circ + \frac{\beta}{2}$, which is equivalent to $\angle KLC = \frac{\beta}{2}$.

Let I be the incenter of triangle ABC and let D be the tangency point of the incircle with AB. Since $CK \parallel IB$ and $CL \parallel IA$, it holds that $\angle KCL = \angle AIB$. Moreover, from $CN = AD = \frac{b+c-a}{2}$ and $\angle KCN = \frac{\beta}{2}$ we obtain $CK = CN \cos\frac{\beta}{2} = AD \cos\frac{\beta}{2} = AI \cos\frac{\alpha}{2}\cos\frac{\beta}{2}$ and analogously $CL = BI \cos\frac{\alpha}{2}\cos\frac{\beta}{2}$, which imply $\frac{CK}{CL} = \frac{AI}{BI}$. Hence the triangles KCL and AIB are similar, and thus $\angle KLC = \angle ABI = \frac{\beta}{2}$.

2. Reducing modulo 11 yields $x^5 + 4^y \equiv 0 \pmod{11}$, where $x^5 \equiv \pm 1 \pmod{11}$, so we also have $4^y \equiv \pm 1 \pmod{11}$. Congruence $4^y \equiv -1 \pmod{11}$ does not hold for any y, whereas $4^y \equiv 1 \pmod{11}$ holds if and only if $5 \mid y$.

Setting $t = 4^{y/5}$, the equation becomes $x^5 + t^5 = A \cdot B = 2013^z$, where $(x,t) = 1$ and $A = x + t$, $B = x^4 - x^3t + x^2t^2 - xt^3 + t^4$. Furthermore, from $B = A(x^3 - 2x^2t + 3xt^2 - 4t^3) + 5t^4$ we deduce $(A, B) = (A, 5t^4) \mid 5$, but $5 \nmid 2013^z$, so we must have $(A, B) = 1$. Therefore $A = a^z$ and $B = b^z$ for some positive integers a and b with $a \cdot b = 2013$.

On the other hand, from $\frac{1}{16}A^4 \le B \le A^4$ (which is a simple consequence of the mean inequality) we obtain $\frac{1}{16}a^4 \le b \le a^4$, i.e. $\frac{1}{16}a^5 \le ab = 2013 \le a^5$. Therefore $5 \le a \le 8$, which is impossible because 2013 has no divisors in the interval $[5, 8]$.

3. It follows from the properties of function f that, for all $x, y, z, a, b > 0$,

$$f(a^2x, aby, b^2z) = bf(a^2x, ay, z) = b\cdot\frac{z}{a^2x}f(z, ay, a^2x) = \frac{bz}{ax}f(z, y, x) = \frac{b}{a}f(x, y, z).$$

We shall choose a and b in such a way that the triple (a^2x, aby, b^2z) is of the form $(1, k, k+1)$ for some k: namely, we take $a = \frac{1}{\sqrt{x}}$ and b satisfying $b^2z - aby = 1$, which upon solving the quadratic equation yields $b = \frac{y+\sqrt{y^2+4xz}}{2z\sqrt{x}}$ and $k = \frac{y(y+\sqrt{y^2+4xz})}{2xz}$. Now we easily obtain

$$f(x, y, z) = \frac{a}{b}f(a^2x, aby, b^2z) = \frac{a}{b}f(1, k, k+1) = \frac{a}{b}(k+1) = \frac{y + \sqrt{y^2 + 4xz}}{2x}.$$

It is directly verified that f satisfies the problem conditions.

4. Consider the graph $\mathcal{G}$ whose vertices are the contestants, where there is an edge between two contestants if and only if they are not friends.

Lemma. There is a vertex in graph $\mathcal{G}$ with degree at most 2.

Proof. Suppose that each vertex has a degree at least three. Consider the longest induced path $P = u_0u_1u_2 \dots u_k$ in the graph (that is, the path in which no two nonadjacent vertices are connected by an edge). The vertex u_0 is connected to another two vertices v and w, which must be outside the path P. Since P is the longest induced path, v and w have neighbors in it. Let u_i and u_j be the neighbors of v and w respectively with the smallest i and j; assume without loss of generality that $i \geq j$. Then $u_0, u_1, \dots, u_i, v$ form a weakly friendly cycle, but w has two neighbors in it (u_0 and u_j), a contradiction.

We now prove the problem statement by induction on the number n of vertices in $\mathcal{G}$. For $n \leq 3$ the statement is trivial; assume that it holds for $n-1$. By the Lemma, there is a vertex v in $\mathcal{G}$ of degree at most two. Graph $\mathcal{G}'$, obtained by removing vertex v (and all edges incident to it), clearly satisfies the problem conditions, so its vertices can be partitioned into three rooms in a desired way. Since v has no neighbors in at least one of the rooms, we can place v in that room, finishing the proof.

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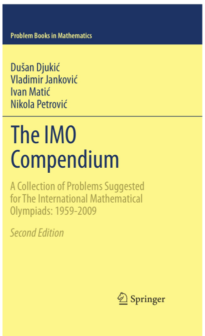

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