Ministry of Education, Science and Technological Development
Serbian Mathematical Society

SERBIAN MATHEMATICAL OLYMPIAD FOR HIGH SCHOOL STUDENTS

Novi Sad – April 5, 2013

First Day

1. Let k be a fixed natural number. A bijection $f : \mathbb{Z} \to \mathbb{Z}$ is such that if i and j are any integers satisfying $|i - j| \le k$, then $|f(i) - f(j)| \le k$. Prove that for any $i, j \in \mathbb{Z}$

$$|f(i) - f(j)| = |i - j|.$$

(Miljan Knežević)

2. Define

$$S_n = \left\{ \binom{n}{n}, \binom{2n}{n}, \binom{3n}{n}, \ldots, \binom{n^2}{n} \right\}, \quad \text{for } n \in \mathbb{N}.$$

a) Prove that there exist infinitely many composite natural numbers n such that S_n *is not* a complete set of residues modulo n.

b) Prove that there exist infinitely many composite natural numbers n such that S_n *is* a complete set of residues modulo n.

(Miloš Milosavljević)

3. Let M, N and P be the midpoints of sides BC, AC and AB respectively, and O be the circumcenter of an acute-angled triangle ABC. The circumcircles of triangles BOC and MNP intersect at distinct points X and Y inside the triangle ABC. Prove that

$$\measuredangle BAX = \measuredangle CAY.$$

(Marko Djikić)

Time allowed: 270 minutes.
Each problem is worth 7 points.

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SERBIAN MATHEMATICAL OLYMPIAD FOR HIGH SCHOOL STUDENTS

Novi Sad – April 6, 2013

Second Day

4. Find all $n \in \mathbb{N}$ for which it is possible to partition the set $\{1, 2, \dots, 3n\}$ into n three-element subsets $\{a, b, c\}$ in which $b - a$ and $c - b$ are different numbers from the set $\{n - 1, n, n + 1\}$.

(Dušan Djukić)

5. Let A' and B' be the feet of the altitudes from A and B respectively of an acute-angled triangle ABC ($AC \neq BC$). Circle k through points A' and B' is tangent to side AB at point D. If the triangles ADA' and BDB' have equal areas, prove that

$$\measuredangle A'DB' = \measuredangle ACB.$$

(Miloš Milosavljević)

6. Determine the largest constant $K \in \mathbb{R}$ with the following property:
If $a_1, a_2, a_3, a_4 > 0$ are such that for any $i, j, k \in \mathbb{N}$, $1 \leqslant i < j < k \leqslant 4$ it holds that $a_i^2 + a_j^2 + a_k^2 \geqslant 2(a_i a_j + a_j a_k + a_k a_i)$, then

$$a_1^2 + a_2^2 + a_3^2 + a_4^2 \geqslant K(a_1 a_2 + a_1 a_3 + a_1 a_4 + a_2 a_3 + a_2 a_4 + a_3 a_4).$$

(Dušan Djukić)

Time allowed: 270 minutes.
Each problem is worth 7 points.