Sunday, May 4, 2014

Problem 1. Let x, y and z be positive real numbers such that $xy + yz + zx = 3xyz$. Prove that

$$x^2y + y^2z + z^2x \geq 2(x + y + z) - 3$$

and determine when equality holds.

Problem 2. A *special number* is a positive integer n for which there exist positive integers a, b, c and d with

$$n = \frac{a^3 + 2b^3}{c^3 + 2d^3}.$$

Prove that:

(a) there are infinitely many special numbers;

(b) 2014 is not a special number.

Problem 3. Let $ABCD$ be a trapezium inscribed in a circle Γ with diameter AB. Let E be the intersection point of the diagonals AC and BD. The circle with center B and radius BE meets Γ at the points K and L, where K is on the same side of AB as C. The line perpendicular to BD at E intersects CD at M.

Prove that KM is perpendicular to DL.

Problem 4. Let n be a positive integer. A regular hexagon with side length n is divided into equilateral triangles with side length 1 by lines parallel to its sides.

Find the number of regular hexagons all of whose vertices are among the vertices of the equilateral triangles.

Language: English

Time: 4 hours and 30 minutes
Each problem is worth 10 points