X International Zhautykov Olympiad in Mathematics
Almaty, 2014

14 January, 2014, 9.00–13.30
First day
(Each problem is worth 7 points)

1. Points *M*, *N*, *K* lie on the sides *BC*, *CA*, *AB* of a triangle *ABC*, respectively and are different from its vertices. The triangle *MNK* is called *beautiful* if $\angle BAC = \angle KMN$ and $\angle ABC = \angle KNM$. If in the triangle *ABC* there are two beautiful triangles with a common vertex, prove that the triangle *ABC* is right-angled.

2. Does there exist a function $f : \mathbf{R} \to \mathbf{R}$ satisfying the following conditions:
(i) for each real y there is a real x such that $f(x) = y$, and
(ii) $f(f(x)) = (x-1)f(x) + 2$ for all real x?

3. Given are 100 different positive integers. We call a pair of numbers *good* if the ratio of these numbers is either 2 or 3. What is the maximum number of good pairs that these 100 numbers can form? (A number can be used in several pairs.)

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4. Does there exist a polynomial $P(x)$ with integral coefficients such that $P\left(1+\sqrt{3}\right)=2+\sqrt{3}$ and $P\left(3+\sqrt{5}\right)=3+\sqrt{5}$?

5. Let $U=\{1, 2,\dots, 2014\}$. For positive integers a, b, c we denote by $f(a, b, c)$ the number of ordered 6-tuples of sets $\left(X_1, X_2, X_3, Y_1, Y_2, Y_3\right)$ satisfying the following conditions:
(i) $Y_1 \subseteq X_1 \subseteq U$ and $\left|X_1\right| = a$;
(ii) $Y_2 \subseteq X_2 \subseteq U \setminus Y_1$ and $\left|X_2\right| = b$;
(iii) $Y_3 \subseteq X_3 \subseteq U \setminus \left(Y_1 \cup Y_2\right)$ and $\left|X_3\right| = c$.
Prove that $f(a,b,c)$ does not change when a, b, c are rearranged.

6. Four segments divide a convex quadrilateral into nine quadrilaterals. The points of intersections of these segments lie on the diagonals of the quadrilateral (see figure). It is known that the quadrilaterals 1, 2, 3, 4 admit inscribed circles. Prove that the quadrilateral 5 also has an inscribed circle.