The 7^{th} Romanian Master of Mathematics Competition

Day 1: Friday, February 27, 2015, Bucharest

Language: English

Problem 1. Does there exist an infinite sequence of positive integers a_1, a_2, a_3, … such that a_m and a_n are coprime if and only if $|m - n| = 1$?

Problem 2. For an integer $n \geq 5$, two players play the following game on a regular n-gon. Initially, three consecutive vertices are chosen, and one counter is placed on each. A move consists of one player sliding one counter along any number of edges to another vertex of the n-gon without jumping over another counter. A move is *legal* if the area of the triangle formed by the counters is strictly greater after the move than before. The players take turns to make legal moves, and if a player cannot make a legal move, that player loses. For which values of n does the player making the first move have a winning strategy?

Problem 3. A finite list of rational numbers is written on a blackboard. In an *operation*, we choose any two numbers a, b, erase them, and write down one of the numbers

$$a + b,\ a - b,\ b - a,\ a \times b,\ a/b \text{ (if } b \neq 0),\ b/a \text{ (if } a \neq 0).$$

Prove that, for every integer $n > 100$, there are only finitely many integers $k \geq 0$, such that, starting from the list

$$k + 1,\ k + 2,\ \ldots,\ k + n,$$

it is possible to obtain, after $n - 1$ operations, the value $n!$.

Each of the three problems is worth 7 points.
Time allowed $4\frac{1}{2}$ hours.

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Day 2: Saturday, February 28, 2015, Bucharest

Language: English

Problem 4. Let ABC be a triangle, and let D be the point where the incircle meets side BC. Let J_b and J_c be the incentres of the triangles ABD and ACD, respectively. Prove that the circumcentre of the triangle AJ_bJ_c lies on the angle bisector of $\angle BAC$.

Problem 5. Let $p \geq 5$ be a prime number. For a positive integer k, let $R(k)$ be the remainder when k is divided by p, with $0 \leq R(k) \leq p-1$. Determine all positive integers $a < p$ such that, for every $m = 1, 2, \ldots, p-1$,

$$m + R(ma) > a.$$

Problem 6. Given a positive integer n, determine the largest real number μ satisfying the following condition: for every set C of $4n$ points in the interior of the unit square U, there exists a rectangle T contained in U such that

- the sides of T are parallel to the sides of U;
- the interior of T contains exactly one point of C;
- the area of T is at least μ.

Each of the three problems is worth 7 points.
Time allowed $4\frac{1}{2}$ hours.