The 7^{th} Romanian Master of Mathematics Competition

Solutions for the Day 1

Problem 1. Does there exist an infinite sequence of positive integers a_1, a_2, a_3, $\ldots$ such that a_m and a_n are coprime if and only if $|m - n| = 1$?

(Peru) Jorge Tipe

Solution. The answer is in the affirmative.

The idea is to consider a sequence of pairwise distinct primes p_1, p_2, p_3, $\ldots$, cover the positive integers by a sequence of finite non-empty sets I_n such that I_m and I_n are disjoint if and only if m and n are one unit apart, and set $a_n = \prod_{i \in I_n} p_i$, $n = 1, 2, 3, \ldots$.

One possible way of finding such sets is the following. For all positive integers n, let

$$2n \in I_k \qquad \text{for all } k = n, n+3, n+5, n+7, \ldots; \qquad \text{and}$$
$$2n - 1 \in I_k \qquad \text{for all } k = n, n+2, n+4, n+6, \ldots.$$

Clearly, each I_k is finite, since it contains none of the numbers greater than $2k$. Next, the number p_{2n} ensures that I_n has a common element with each I_{n+2i}, while the number p_{2n-1} ensures that I_n has a common element with each I_{n+2i+1} for $i = 1, 2, \ldots$. Finally, none of the indices appears in two consecutive sets.

Remark. The sets I_n from the solution above can explicitly be written as

$$I_n = \{2n - 4k - 1 \colon k = 0, 1, \ldots, \lfloor (n-1)/2 \rfloor\} \cup \{2n - 4k - 2 \colon k = 1, 2, \ldots, \lfloor n/2 \rfloor - 1\} \cup \{2n\},$$

The above construction can alternatively be described as follows: Let p_1, p_1', p_2, p_2', $\ldots$, p_n, p_n', $\ldots$ be a sequence of pairwise distinct primes. With the standard convention that empty products are 1, let

$$P_n = \begin{cases} p_1 p_2' p_3 p_4' \cdots p_{n-4} p_{n-3}' p_{n-2}, & \text{if } n \text{ is odd,} \\ p_1' p_2 p_3' p_4 \cdots p_{n-3}' p_{n-2}, & \text{if } n \text{ is even,} \end{cases}$$

and define $a_n = P_n p_n p_n'$.

Problem 2. For an integer $n \geq 5$, two players play the following game on a regular n-gon. Initially, three consecutive vertices are chosen, and one counter is placed on each. A move consists of one player sliding one counter along any number of edges to another vertex of the n-gon without jumping over another counter. A move is *legal* if the area of the triangle formed by the counters is strictly greater after the move than before. The players take turns to make legal moves, and if a player cannot make a legal move, that player loses. For which values of n does the player making the first move have a winning strategy?

(UNITED KINGDOM) JEREMY KING

Solution. We shall prove that the first player wins if and only the exponent of 2 in the prime decomposition of $n-3$ is odd.

Since the game is identical for both players, has finitely many possible states and always terminates, we can label the possible states Wins od Losses according as whether a player faced with that position has a winning strategy or not. A state is a Win if and only if there is some legal move taking the state to a Loss, and a state is a Loss if and only if all moves take that state to a Win (including the case where there are no legal moves).

Lemma. *Any configuration in which the triangle formed by the three counters is not isosceles is necessarily a Win.*

Proof. Label the positions of the counters X, Y, Z so that the arc YZ of the circumcircle is shortest and the arc ZX is longest. Begin by moving the counter at Z around the polygon on the arc YZX until it forms an isosceles triangle XYZ' with apex at Y (note that the arc XY is less than half the circle, so that Z does not jump over the counter at X). If this configuration is a Loss, we are done.

If instead this configuration is a Win, then the counters can be moved legally from triangle XYZ' to reach a losing state. This cannot involve the counter at Y, so by symmetry a Loss state can be reached by moving the counter at Z' to a new location Z''. But then the counter at Z could have been moved to Z'' in the first place, so the original configuration was a Win as well. $\square$

For every nonzero integer x, denote by $v_2(x)$ the exponent of 2 in the prime decomposition of x. Now, given a configuration in which the triangle formed by the three counters is isosceles, the arcs between the vertices having lengths a, a, b respectively (in appropriate units so that $2a+b=n$), we show that the configuration is a Win if and only if $a \neq b$ and $v_2(a-b)$ is odd.

Write $b = a \pm |a-b|$ and notice that the only other isosceles triangle that can be reached from the original configuration is one with arc lengths a, $a \pm |a-b|/2$, $a \pm |a-b|/2$. If $|a-b|$ is odd, this is of course impossible, so the configuration is a Loss, since all non-isosceles configurations are Wins, by the lemma.

If instead $|a-b|$ is even, then all states that can be reached from the original configuration are Wins, except possibly the state with arc lengths a, $a \pm |a-b|/2$, $a \pm |a-b|/2$. Consequently, (a,a,b) is a Win if and only if $(a, a \pm |a-b|/2, a \pm |a-b|/2)$ is a Loss. Since the side lengths of this new triangle differ by $|a-b|/2$, the conclusion follows inductively once the exceptional and trivial case $a=b$ is dealt with.

As an immediate corollary, the configuration with arc lengths 1, 1, $n-2$ (the starting configuration of the question) is a Win if and only if $v_2(n-3)$ is odd.

Remark. Relying on the solution presented above, one may also derive an explicit winning strategy. Denote the position in the game by the multiset $\{a, b, c\}$ of thr lengths of the three arcs between the tokens (again in appropriate units so that $a+b+c=n$). A move now consists in choosing two of the three numbers a, b, c, and replacing them by two numbers with the same sum so as to strictly increase the minimum of the pair.

The winning strategy for a player is to obtain at the end of each of his moves the positions of the form $\{a, a, b\}$, where $a = b$ or $v_2(a - b)$ is even; we say that such position is *good*. At the beginning of the game, the position is good exactly if $v_2(n - 3)$ is even.

Now, there is at most one position of the form $\{a', a', b'\}$ which may be obtained by a move from a good position $\{a, a, b\}$ — that is, with $b' = a$. This position is not good, thus it suffices to show that it is possible to obtain a good position from any non-good one by a move.

Let now $\{a, b, c\}$ be a non-good position, with $a \leq b \leq c$. If $a+c = 2b$ then one may get the good position (b, b, b). Assume now that $a + c \neq 2b$. If $v_2(c + a - 2b)$ is even, then it is possible to achieve the good position $\{b, b, c + a - b\}$; otherwise, $c + a$ is necessarily even, and one may get the good position $\{(c + a)/2, (c + a)/2, b\}$.

Problem 3. A finite list of rational numbers is written on a blackboard. In an *operation*, we choose any two numbers a, b, erase them, and write down one of the numbers

$$a+b,\ a-b,\ b-a,\ a\times b,\ a/b \text{ (if } b\neq 0),\ b/a \text{ (if } a\neq 0).$$

Prove that, for every integer $n > 100$, there are only finitely many integers $k \geq 0$, such that, starting from the list

$$k+1,\ k+2,\ \ldots,\ k+n,$$

it is possible to obtain, after $n-1$ operations, the value $n!$.

(United Kingdom) Alexander Betts

Solution. We prove the problem statement even for all positive integer n.

There are only finitely many ways of constructing a number from n pairwise distinct numbers $x_1, \ldots, x_n$ only using the four elementary arithmetic operations, and each x_k exactly once. Each such formula for $k > 1$ is obtained by an elementary operation from two such formulas on two disjoint sets of the x_i.

A straightforward induction on n shows that the outcome of each such construction is a number of the form

$$\frac{\displaystyle\sum_{\alpha_1,\ldots,\alpha_n\in\{0,1\}} a_{\alpha_1,\ldots,\alpha_n} x_1^{\alpha_1}\cdots x_n^{\alpha_n}}{\displaystyle\sum_{\alpha_1,\ldots,\alpha_n\in\{0,1\}} b_{\alpha_1,\ldots,\alpha_n} x_1^{\alpha_1}\cdots x_n^{\alpha_n}}, \tag{$*$}$$

where the $a_{\alpha_1,\ldots,\alpha_n}$ and $b_{\alpha_1,\ldots,\alpha_n}$ are all in the set $\{0,\pm1\}$, not all zero of course, $a_{0,\ldots,0} = b_{1,\ldots,1} = 0$, and also $a_{\alpha_1,\ldots,\alpha_n}\cdot b_{\alpha_1,\ldots,\alpha_n} = 0$ for every set of indices.

Since $|a_{\alpha_1,\ldots,\alpha_n}| \leq 1$, and $a_{0,0,\ldots,0} = 0$, the absolute value of the numerator does not exceed $(1+|x_1|)\cdots(1+|x_n|)-1$; in particular, if c is an integer in the range $-n, \ldots, -1$, and $x_k = c+k$, $k = 1, \ldots, n$, then the absolute value of the numerator is at most $(-c)!(n+c+1)!-1 \leq n!-1 < n!$.

Consider now the integral polynomials,

$$P = \sum_{\alpha_1,\ldots,\alpha_n\in\{0,1\}} a_{\alpha_1,\ldots,\alpha_n}(X+1)^{\alpha_1}\cdots(X+n)^{\alpha_n},$$

and

$$Q = \sum_{\alpha_1,\ldots,\alpha_n\in\{0,1\}} b_{\alpha_1,\ldots,\alpha_n}(X+1)^{\alpha_1}\cdots(X+n)^{\alpha_n},$$

where the $a_{\alpha_1,\ldots,\alpha_n}$ and $b_{\alpha_1,\ldots,\alpha_n}$ are all in the set $\{0,\pm1\}$, not all zero, $a_{\alpha_1,\ldots,\alpha_n}b_{\alpha_1,\ldots,\alpha_n} = 0$ for every set of indices, and $a_{0,\ldots,0} = b_{1,\ldots,1} = 0$. By the preceding, $|P(c)| < n!$ for every integer c in the range $-n, \ldots, -1$; and since $b_{1,\ldots,1} = 0$, the degree of Q is less than n.

Since every non-zero polynomial has only finitely many roots, and the number of roots does not exceed the degree, to complete the proof it is sufficient to show that the polynomial $P - n!Q$ does not vanish identically, provided that Q does not (which is the case in the problem).

Suppose, if possible, that $P = n!Q$, where $Q \neq 0$. Since $\deg Q < n$, it follows that $\deg P < n$ as well, and since $P \neq 0$, the number of roots of P does not exceed $\deg P < n$, so $P(c) \neq 0$ for some integer c in the range $-n, \ldots, -1$. By the preceding, $|P(c)|$ is consequently a positive integer less than $n!$. On the other hand, $|P(c)| = n!|Q(c)|$ is an integral multiple of $n!$. A contradiction.

Remark. Alternatively, it can be shown by induction on n that

$$\max(|P(c)|,\ 2|Q(c)|) \leq \prod_{k=1}^{n}\max(|c+k|, 2),$$

for all integers c. In case $n > 8$, this provides a solution along the same lines.

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Solutions for the Day 2

Problem 4. Let ABC be a triangle, let D be the touchpoint of the side BC and the incircle of the triangle ABC, and let J_b and J_c be the incentres of the triangles ABD and ACD, respectively. Prove that the circumcentre of the triangle AJ_bJ_c lies on the bisectrix of the angle BAC.

(Russia) Fedor Ivlev

Solution. Let the incircle of the triangle ABC meet CA and AB at points E and F, respectively. Let the incircles of the triangles ABD and ACD meet AD at points X and Y, respectively. Then $2DX = DA+DB-AB = DA+DB-BF-AF = DA-AF$; similarly, $2DY = DA-AE = 2DX$. Hence the points X and Y coincide, so $J_bJ_c \perp AD$.

Now let O be the circumcentre of the triangle AJ_bJ_c. Then $\angle J_bAO = \pi/2 - \angle AOJ_b/2 = \pi/2 - \angle AJ_cJ_b = \angle XAJ_c = \frac{1}{2}\angle DAC$. Therefore, $\angle BAO = \angle BAJ_b + \angle J_bAO = \frac{1}{2}\angle BAD + \frac{1}{2}\angle DAC = \frac{1}{2}\angle BAC$, and the conclusion follows.

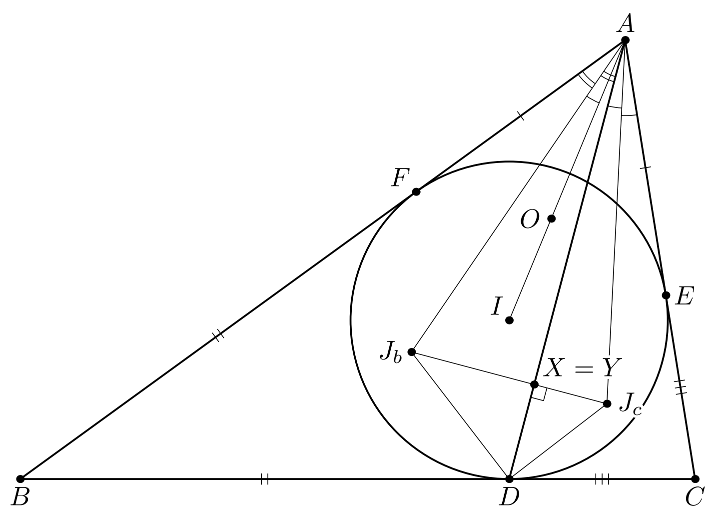

Fig. 1

Problem 5. Let $p \geq 5$ be a prime number. For a positive integer k we denote by $R(k)$ the remainder of k when divided by p. Determine all positive integers $a < p$ such that

$$m + R(ma) > a$$

for every $m = 1, 2, \ldots, p-1$.

(Bulgaria) Alexander Ivanov

Solution. The required integers are $p-1$ along with all the numbers of the form $\lfloor p/q \rfloor$, $q = 2, \ldots, p-1$. In other words, these are $p-1$, along with the numbers $1, 2, \ldots, \lfloor \sqrt{p} \rfloor$, and also the (distinct) numbers $\lfloor p/q \rfloor$, $q = 2, \ldots, \left\lfloor \sqrt{p} - \frac{1}{2} \right\rfloor$.

We begin by showing that these numbers satisfy the conditions in the statement. It is readily checked that $p-1$ satisfies the required inequalities, since $m + R(m(p-1)) = m + (p-m) = p > p-1$ for all $m = 1, \ldots, p-1$.

Now, consider any number a of the form $a = \lfloor p/q \rfloor$, where q is an integer greater than 1 but less than p; then $p = aq + r$ with $0 < r < q$. Choose any integer $m \in (0, p)$ and write $m = xq + y$ with $x, y \in \mathbb{Z}$, $0 < y \leq q$ (notice that x is nonnegative). Then

$$R(ma) = R(ay + xaq) = R(ay + xp - xr) = R(ay - xr).$$

Since $ay - xr \leq ay \leq aq < p$, we obtain $R(ay - xr) \geq ay - xr$ and hence

$$m + R(ma) \geq (xq + y) + (ay - xr) = x(q - r) + y(a + 1) \geq a + 1$$

by $q > r$ and $y \geq 1$. Thus a satisfies the required condition.

Finally, we show that if an integer $a \in (0, p-1)$ satisfies the required condition then a is indeed of the form $a = \lfloor p/q \rfloor$ for some integer $q \in (0, p)$. This is clear for $a = 1$, so we may (and will) assume that $a \geq 2$.

Write $p = aq + r$ with $q, r \in \mathbb{Z}$ and $0 < r < a$; since $a \geq 2$ we have $q < p/2$. Choose $m = q + 1 < p$; we have $R(ma) = R(aq + a) = R(p + (a - r)) = a - r$, so

$$a < m + R(ma) = q + 1 + a - r,$$

which yields $r < q+1$. Moreover, if $r = q$, then $p = q(a+1)$ which is impossible by $1 < a+1 < p$. Thus $r < q$, and we have

$$0 \leq \frac{p}{q} - a = \frac{r}{q} < 1,$$

which proves $a = \lfloor p/q \rfloor$.

Problem 6. Given a positive integer n, determine the largest real number μ satisfying the following condition: for every $4n$-point configuration C in an open unit square U, there exists an open rectangle in U, whose sides are parallel to those of U, which contains exactly one point of C, and has an area greater than or equal to μ.

(Bulgaria) Nikolai Beluhov

Solution. The required maximum is $\frac{1}{2n+2}$. To show that the condition in the statement is not met if $\mu > \frac{1}{2n+2}$, let $U = (0,1) \times (0,1)$, choose a small enough positive ϵ, and consider the configuration C consisting of the n four-element clusters of points $\left(\frac{i}{n+1} \pm \epsilon\right) \times \left(\frac{1}{2} \pm \epsilon\right)$, $i = 1, \ldots, n$, the four possible sign combinations being considered for each i. Clearly, every open rectangle in U, whose sides are parallel to those of U, which contains exactly one point of C, has area at most $\left(\frac{1}{n+1} + \epsilon\right) \cdot \left(\frac{1}{2} + \epsilon\right) < \mu$ if ϵ is small enough.

We now show that, given a finite configuration C of points in an open unit square U, there always exists an open rectangle in U, whose sides are parallel to those of U, which contains exactly one point of C, and has an area greater than or equal to $\mu_0 = \dfrac{2}{|C| + 4}$.

To prove this, usage will be made of the following two lemmas whose proofs are left at the end of the solution.

Lemma 1. *Let k be a positive integer, and let $\lambda < \dfrac{1}{\lfloor k/2 \rfloor + 1}$ be a positive real number. If $t_1, \ldots, t_k$ are pairwise distinct points in the open unit interval $(0,1)$, then some t_i is isolated from the other t_j by an open subinterval of $(0,1)$ whose length is greater than or equal to λ.*

Lemma 2. *Given an integer $k \geq 2$ and positive integers $m_1, \ldots, m_k$,*

$$\left\lfloor \frac{m_1}{2} \right\rfloor + \sum_{i=1}^{k} \left\lfloor \frac{m_i}{2} \right\rfloor + \left\lfloor \frac{m_k}{2} \right\rfloor \leq \sum_{i=1}^{k} m_i - k + 2.$$

Back to the problem, let $U = (0,1) \times (0,1)$, project C orthogonally on the x-axis to obtain the points $x_1 < \cdots < x_k$ in the open unit interval $(0,1)$, let ℓ_i be the vertical through x_i, and let $m_i = |C \cap \ell_i|$, $i = 1, \ldots, k$.

Setting $x_0 = 0$ and $x_{k+1} = 1$, assume that $x_{i+1} - x_{i-1} > (\lfloor m_i/2 \rfloor + 1)\mu_0$ for some index i, and apply Lemma 1 to isolate one of the points in $C \cap \ell_i$ from the other ones by an open subinterval $x_i \times J$ of $x_i \times (0,1)$ whose length is greater than or equal to $\mu_0/(x_{i+1} - x_{i-1})$. Consequently, $(x_{i-1}, x_{i+1}) \times J$ is an open rectangle in U, whose sides are parallel to those of U, which contains exactly one point of C and has an area greater than or equal to μ_0.

Next, we rule out the case $x_{i+1} - x_{i-1} \leq (\lfloor m_i/2 \rfloor + 1)\mu_0$ for all indices i. If this were the case, notice that necessarily $k > 1$; also, $x_1 - x_0 < x_2 - x_0 \leq (\lfloor m_1/2 \rfloor + 1)\mu_0$ and $x_{k+1} - x_k < x_{k+1} - x_{k-1} \leq (\lfloor m_k/2 \rfloor + 1)\mu_0$. With reference to Lemma 2, write

$$\begin{aligned}
2 = 2(x_{k+1} - x_0) &= (x_1 - x_0) + \sum_{i=1}^{k} (x_{i+1} - x_{i-1}) + (x_{k+1} - x_k) \\
&< \left(\left(\left\lfloor \frac{m_1}{2} \right\rfloor + 1 \right) + \sum_{i=1}^{k} \left(\left\lfloor \frac{m_i}{2} \right\rfloor + 1 \right) + \left(\left\lfloor \frac{m_k}{2} \right\rfloor + 1 \right) \right) \cdot \mu_0 \\
&\leq \left(\sum_{i=1}^{k} m_i + 4 \right) \mu_0 = (|C| + 4)\mu_0 = 2,
\end{aligned}$$

and thereby reach a contradiction.

Finally, we prove the two lemmas.

Proof of Lemma 1. Suppose, if possible, that no t_i is isolated from the other t_j by an open subinterval of $(0,1)$ whose length is greater than or equal to λ. Without loss of generality, we may (and will) assume that $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = 1$. Since the open interval (t_{i-1}, t_{i+1}) isolates t_i from the other t_j, its length, $t_{i+1} - t_{i-1}$, is less than λ. Consequently, if k is odd we have $1 = \sum_{i=0}^{(k-1)/2}(t_{2i+2} - t_{2i}) < \lambda\big(1 + \frac{k-1}{2}\big) < 1$; if k is even, we have $1 < 1 + t_k - t_{k-1} = \sum_{i=0}^{k/2-1}(t_{2i+2} - t_{2i}) + (t_{k+1} - t_{k-1}) < \lambda\big(1 + \frac{k}{2}\big) < 1$. A contradiction in either case.

Proof of Lemma 2. Let I_0, respectively I_1, be the set of all indices i in the range 2, …, $k-1$ such that m_i is even, respectively odd. Clearly, I_0 and I_1 form a partition of that range. Since $m_i \geq 2$ if i is in I_0, and $m_i \geq 1$ if i is in I_1 (recall that the m_i are positive integers),

$$\sum_{i=2}^{k-1} m_i = \sum_{i\in I_0} m_i + \sum_{i\in I_1} m_i \geq 2|I_0| + |I_1| = 2(k-2) - |I_1|, \quad \text{or} \quad |I_1| \geq 2(k-2) - \sum_{i=2}^{k-1} m_i.$$

Therefore,

$$\begin{aligned}
\left\lfloor \frac{m_1}{2} \right\rfloor + \sum_{i=1}^{k} \left\lfloor \frac{m_i}{2} \right\rfloor + \left\lfloor \frac{m_k}{2} \right\rfloor &\leq m_1 + \left(\sum_{i=2}^{k-1} \frac{m_i}{2} - \frac{|I_1|}{2} \right) + m_k \\
&\leq m_1 + \left(\frac{1}{2} \sum_{i=2}^{k-1} m_i - (k-2) + \frac{1}{2} \sum_{i=2}^{k-1} m_i \right) + m_k \\
&= \sum_{i=1}^{k} m_i - k + 2.
\end{aligned}$$

□

Remark. In case $4n$ is replaced by a positive integer k not divisible by 4, we do not yet know the maximal μ satisfying the corresponding condition.