Ministry of Education, Science and Technological Development
Serbian Mathematical Society

9th SERBIAN MATHEMATICAL OLYMPIAD FOR HIGH SCHOOL STUDENTS

Belgrade – March 27, 2015

First Day

1. Let $ABCD$ be an inscribed quadrilateral and let M, N, P and Q be the midpoints of the sides DA, AB, BC and CD, respectively. The diagonals AC and BD intersect at point E, and the circumcircles of $\triangle EMN$ and $\triangle EPQ$ meet at point $F \neq E$. Prove that $EF \perp AC$. *(Dušan Djukić)*

2. A natural number k is given. For $n \in \mathbb{N}$ we define $f_k(n)$ as the smallest integer greater than kn such that $nf_k(n)$ is a perfect square. Prove that $f_k(m) = f_k(n)$ implies $m = n$. *(Nikola Petrović)*

3. A guard proposes the following game to the prisoners. All prisoners are to be taken to the prison yard, where each of them will be put a hat in one of 5 possible colors onto his head, and aligned so that each of them can see all hats but his own. The guard will then ask the first prisoner to say aloud whether he knows the color of his hat. If he answers "no", he will be publicly executed. Otherwise, he will be asked to say the color of his hat in such a way that others do not hear his answer. If the answer is correct, he will be freed, otherwise he will be publicly executed. The guard will then go on to the next prisoner in line and repeat the procedure, and so on. The prisoners may devise a strategy before the game starts, but no communication between them during the game is allowed. If there are 2015 prisoners, what is the maximal number of them that can have guaranteed freedom using an optimal strategy? *(Bojan Bašić)*

Time allowed: 270 minutes.
Each problem is worth 7 points.

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Second Day

4. For a nonzero integer a, denote by $v_2(a)$ the largest nonnegative integer k such that $2^k \mid a$. Given $n \in \mathbb{N}$, determine the largest possible cardinality of a subset A of set $\{1, 2, 3, \ldots, 2^n\}$ with the following property:

for all $x, y \in A$ with $x \neq y$, $v_2(x - y)$ is even.

(Dušan Djukić)

5. Prove that the inequality

$$\frac{x-y}{xy+2y+1} + \frac{y-z}{yz+2z+1} + \frac{z-x}{zx+2x+1} \geqslant 0$$

holds for any nonnegative real numbers x, y and z. *(Dušan Djukić)*

6. Find all nonnegative integer solutions of the equation

$$(2^{2015} + 1)^x + 2^{2015} = 2^y + 1.$$

(Bojan Bašić)

Time allowed: 270 minutes.
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