The 1st International Olympiad of Metropolises

September 2016

Solutions of day 1

Problem 1. Find all positive integers n such that there exist n consecutive positive integers whose sum is a perfect square. *(Pavel Kozhevnikov)*

Answer: $n = 2^s m$, where m is any odd integer, and s is either 0 or odd.

Let $S(n,t) = (t+1) + (t+2) + \ldots + (t+n) = (2t+n+1)n/2$.

For odd n one may put $t = (n-1)/2$ and obtain $S(n,t) = n^2$.

Let n be even, $n = 2^s m$, where s is a positive integer, and m is odd. It follows that $2t+n+1$ is odd. Hence 2^{s-1} divides $S(n,t)$, while 2^s does not. This means that for even s the answer is negative. For odd s one may put $t = (mx^2 - n - 1)/2$ for some odd $x > n$ and obtain $S(n,t) = 2^{s-1}m^2x^2$. $\square$

Problem 2. Let $a_1, \ldots, a_n$ be positive integers satisfying the inequality

$$\sum_{i=1}^{n} \frac{1}{a_i} \le \frac{1}{2}.$$

Every year, the government of Optimistica publishes its *Annual Report* with n economic indicators. For each $i = 1, \ldots, n$, the possible values of the i-th indicator are $1, 2, \ldots, a_i$. The Annual Report is said to be *optimistic* if at least $n-1$ indicators have higher values than in the previous report. Prove that the government can publish optimistic Annual Reports in an infinitely long sequence.

(Ivan Mitrofanov, Fedor Petrov)

First we replace each a_i by a power of 2. For every $1 \le i \le n$, let k_i be the positive integer that satisfies $2^{k_i} \le a_i < 2^{k_i+1}$. Notice that $\sum\limits_{i=1}^{n} \frac{1}{2^{k_i}} < \frac{2}{a_i} \le 1$.

For every $1 \le i \le n$, we will choose a residue class A_i modulo 2^{k_i} in such a way that the classes $A_1, \ldots, A_n$ are pairwise disjoint. Without loss of generality we can assume that $k_1 \le k_2 \le \ldots \le k_n$. We choose $A_1, A_2, \ldots, A_n$ in this order. The residue class A_1 can be chosen arbitrarily. Suppose that we have already chosen the classes $A_1, \ldots, A_{i-1}$. In order to find the next class A_i, we require a residue modulo 2^{k_i} which is not used in any of $A_1, \ldots, A_{i-1}$. Notice that for each $j < i$, the set A_j is the union of $2^{k_i - k_j}$ different residue classes modulo 2^{k_i}. As $\sum\limits_{j=1}^{i-1} 2^{k_i-k_j} < 2^{k_i} \sum\limits_{j=1}^{n} 2^{-k_j} < 2^{k_i}$, there are unused residues modulo 2^{k_i} which makes it possible to choose the new class A_i.

Now let us turn to the solution of the problem. For every $1 \le i \le n$, we will use only the first 2^{k_i} values of the i-th indicator. In the beginning let all indicators be equal to 1. In the y-th year, let the i-th indicator drop to 1 if $y \in A_i$, otherwise let the indicator increase by 1. Notice that the i-th indicator increases at most $2^{k_i} - 1$ times in a row, then drops to 1, so it never exceeds the bound $2^{k_i} \le a_i$ and therefore the values of the indicator form a valid report in every year. Since the residue classes $A_1, \ldots, A_n$ are pairwise disjoint, at most one indicator drops in the same year, the reports keep optimistic. □

Problem 3. Let $A_1A_2 \ldots A_n$ be a cyclic convex polygon whose circumcenter is strictly in its interior. Let B_1, B_2, …, B_n be arbitrary points on the sides A_1A_2, A_2A_3, …, A_nA_1, respectively, other than the vertices. Prove that

$$\frac{B_1B_2}{A_1A_3} + \frac{B_2B_3}{A_2A_4} + \ldots + \frac{B_nB_1}{A_nA_2} > 1\,.$$

(Nairi Sedrakyan, David Harutyunyan)

Lemma 1. Suppose that a triangle without obtuse angle is inscribed in a circle of radius R. Then the perimeter of the triangle is greater than $4R$.
Proof. Let ABC be our triangle.
Assume that triangle ABC is right. Without loss of generality $\angle B = 90^\circ$ and $AC = 2R$. Then $AB + BC + AC > AC + AC = 4R$.
Assume that triangle ABC is acute. Let K, L, M be the midpoints of the sides AB, BC, AC respectively. The point O is the orthocentre of the triangle KLM, which is acute as well as the similar triangle ABC. Thus O lies inside the triangle KLM. Let line MO intersect the segment KL at the point P. We have $AB + BC + AC = 2(AK + KL + LC) = 2(AK + KP) + 2(PL + LC) > 2AP + 2PC > 2AO + 2CO = 4R$ (the last inequality uses that the angles $\angle AOP$ and $\angle COP$ are obtuse). *Lemma 1 is proved.*

Lemma 2. Assume that a polygon is inscribed in a circle of radius R, and the center of the circle lies inside the polygon. Then the perimeter P of the polygon is greater than $4R$.
Proof. Let $A_1A_2 \ldots A_n$ be our polygon. The diagonals $A_1A_3, A_1A_4, \ldots, A_1A_{n-1}$ partition it into $n-2$ triangles. The point O belongs to the interior or the boundary of $A_1A_iA_{i+1}$. Now Lemma 2 follows from the Lemma 1:

$$P = (A_1A_2 + \ldots + A_{i-1}A_i) + A_iA_{i+1} + (A_{i+1}A_{i+2} + \ldots + A_nA_1) \ge$$
$$\ge A_1A_i + A_iA_{i+1} + A_{i+1}A_1 > 4R\,.$$

Lemma 2 is proved.

Let us return to the problem. Let R denote the circumradius of the circle $A_1A_2\dots A_n$, let R_i denote the circumradius of $B_iA_{i+1}B_{i+1}$ (further we suppose $A_{n+1} \equiv A_1, A_{n+2} \equiv A_2, B_{n+1} \equiv B_1$). The sine law yields $\frac{B_iB_{i+1}}{\sin\angle A_{i+1}} = 2R_i$, $\frac{A_iA_{i+2}}{\sin\angle A_{i+1}} = 2R$, thus $\frac{B_iB_{i+1}}{A_iA_{i+2}} = \frac{2R_i\sin\angle A_{i+1}}{2R\sin\angle A_{i+1}} = \frac{R_i}{R}$.

$$\frac{B_1B_2}{A_1A_3} + \frac{B_2B_3}{A_2A_4} + \ldots + \frac{B_nB_1}{A_nA_2} > 1$$
$$\Updownarrow$$
$$\frac{R_1}{R} + \frac{R_2}{R} + \ldots + \frac{R_n}{R} > 1$$
$$\Updownarrow$$
$$R_1 + R_2 + \ldots + R_n > R\,.$$

In the triangle $B_iA_{i+1}B_{i+1}$ no side can be greater than the diameter of the circumcircle, therefore $B_iA_{i+1} + A_{i+1}B_{i+1} \le 2R_i + 2R_i = 4R_i$ and $R_i \ge (B_iA_{i+1} + A_{i+1}B_{i+1})/4$. Hence it suffices to prove that

$$R < \frac{B_1A_2 + A_2B_2}{4} + \frac{B_2A_3 + A_3B_3}{4} + \ldots + \frac{B_nA_1 + A_1B_1}{4} = \frac{P}{4},$$

but this follows from Lemma 2. □

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Solutions of day 2

Problem 4. A convex quadrilateral $ABCD$ has right angles at A and C. A point E lies on the extension of the side AD beyond D so that $\angle ABE = \angle ADC$. The point K is symmetric to the point C with respect to point A. Prove that $\angle ADB = \angle AKE$.

(Boyan Obukhov and Fedor Petrov)

The quadrilateral $ABCD$ is inscribed in the circle with diameter BD. Thus $\angle ADB = \angle ACB$ since both angles are subtended by the same arc. So, we have to prove that the angles BCA and AKE are equal, which in turn is equivalent to the claim that the lines BC and KE are parallel.

Note that $\angle BCD + \angle CDA = \angle BAD + \angle ABE < 180°$. It implies that the rays CB and DA have a common point which we denote by F. We have $\angle BFA = 90° - \angle ADC = 90° - \angle ABE = \angle BEA$. So BA is the altitude of the isosceles triangle FBE, this yields $FA = AE$. On the other hand $CA = AK$. So, the diagonals of the quadrilateral $FCEK$ have a common midpoint, i.e., $FCEK$ is a parallelogram. Therefore the lines FC and KE are indeed parallel as desired. □

Problem 5. Let $r(x)$ be a polynomial of odd degree with real coefficients. Prove that there exist only finitely many pairs of polynomials $p(x)$ and $q(x)$ with real coefficients satisfying the equation $(p(x))^3 + q(x^2) = r(x)$. *(Fedor Petrov)*

By replacing x by $-x$ and taking difference, we get $(p(x))^3 - (p(-x))^3 = r(x) - r(-x) = u(x)$, the polynomial $u(x)$ is non-zero, odd, and has the same degree as $r(x)$. We see that $p(x) - p(-x)$ is an odd divisor of $u(x)$. There are only finitely many divisors of $u(x)$ up to a constant factor. So, it suffices to check that for any fixed odd divisor $xa_0(x^2)$ of $u(x)$ there are only finitely many $p(x)$ such that $p(x) - p(-x)$ is proportional to $xa_0(x^2)$, i. e., $p(x)$ is of the form $\lambda xa_0(x^2) + b(x^2)$, where $\lambda \neq 0$ is some unknown constant and $b(t)$ is some unknown polynomial. For proving finiteness we may fix also the sign of λ. We have

$$u(x) = (p(x))^3 - (p(-x))^3 = 2xa_0(x^2) \cdot \left(3\lambda b^2(x^2) + \lambda^3 x^2 a_0^2(x^2)\right).$$

So, the polynomial $3\lambda b^2(t) + \lambda^3 t a_0^2(t)$ (we denoted $t = x^2$) is fixed: $3\lambda b^2(t) + \lambda^3 t a_0^2(t) = 3\lambda_0 b_0^2(t) + \lambda_0^3 t a_0^2(t)$ for some fixed solution $(\lambda_0, b_0(t))$. Rewrite it as

$$\lambda b^2 - \lambda_0 b_0^2 = \frac{\lambda_0^3 - \lambda^3}{3} t a_0^2(t)\,.$$

Dividing by λ_0 and factorizing the LHS as a difference of squares (which is possible in real numbers since λ and λ_0 have the same sign) we see that the pair of polynomials $\sqrt{\lambda/\lambda_0}\,b(t) \pm b_0(t)$ have the form $f(t)$, $\frac{\lambda_0^3-\lambda^3}{3\lambda_0} g(t)$ with $f(t)\cdot g(t) = t a_0^2(t)$. Again we may consider the case when $f(t)$ and $g(t)$ are fixed up to a constant factor: $f(t) = \tau f_0(t)$, $g(t) = \tau^{-1} g_0(t)$. We get

$$2b_0(t) = f(t) - \frac{\lambda_0^3 - \lambda^3}{3\lambda_0} g(t) = \tau f_0(t) - \tau^{-1}\frac{\lambda_0^3 - \lambda^3}{3\lambda_0} g_0(t)\,.$$

If this happens for two different pairs of values (τ, λ) and (τ', λ'), we may take the difference:

$$0 = (\tau - \tau') f_0(t) - \left(\tau^{-1}\frac{\lambda_0^3 - \lambda^3}{3\lambda_0} - (\tau')^{-1}\frac{\lambda_0^3 - (\lambda')^3}{3\lambda_0}\right) g_0(t)\,. \qquad (1)$$

If $\tau \neq \tau'$, it follows that $f_0(t)$ and $g_0(t)$ are proportional; but this is impossible, since their product $f_0(t)\cdot g_0(t) = t a_0^2(t)$ has odd degree. Otherwise, the coefficient of $f(t)$ in (1) is zero, hence coefficient of $g(t)$ is also zero, from which we obtain $(\lambda')^3 = \lambda^3$. It means that τ and λ are fixed, hence $f(t)$ and $g(t)$ are fixed, and there is at most one solution. Since on each step we diverged into finite number of cases, there is no more than a finite number of solutions totally. □

Problem 6. In a country with n cities, some pairs of cities are connected by one-way flights operated by one of two companies A and B. Two cities can be connected by more than one flight in either direction. An AB-word w is called *implementable* if there is a sequence of connected flights whose companies' names form the word w. Given that every AB-word of length 2^n is implementable, prove that every finite AB-word is implementable. (An AB-word of length k is an arbitrary sequence of k letters A or B; e. g. $AABA$ is a word of length 4.) *(Ivan Mitrofanov)*

Assume the contrary. Then there exist non-implementable words. Let $w = a_1 a_2 \ldots a_N$ be the shortest (or one of the shortest) non-implementable word. It is clear that $N > 2^n$. For any integer $0 \leq i \leq N$ denote by A_i the set of all cities that are the possible terminals of sequences of flights, that correspond to the word $a_1 a_2 \ldots a_i$. The set A_0 consists of all cities, A_N is empty. Since there are 2^n different subsets of the set of all cities, it follows by the pigeonhole principle that $A_i = A_j$ for some $i < j$.

Consider the word $w' = a_1 a_2 \ldots a_{i-1} a_i a_{j+1} a_{j+2} \ldots a_N$. Since it is shorter than w, we have that it is implementable. Let S be a sequence of flights implementing w'. By S_1 denote the sequence of the first i flights of S, by S_2 denote the sequence of the last $N-j$ flights of S, by T denote the endpoint of S_1. By construction, $T \in A_i$. Then, since $A_i = A_j$, it follows that there exists a sequence of flights S_3 implementing $a_1 a_2 \ldots a_j$ and T is its terminal city.

But then the sequence of flights $S_3 S_2$ corresponds to $w = a_1 a_2 \ldots a_N$ and w is implementable. This contradiction proves the statement of the problem. $\square$