The 8^{th} Romanian Master of Mathematics Competition

Day 1: Friday, February 26, 2016, Bucharest

Language: English

Problem 1. Let ABC be a triangle and let D be a point on the segment BC, $D \neq B$ and $D \neq C$. The circle ABD meets the segment AC again at an interior point E. The circle ACD meets the segment AB again at an interior point F. Let A' be the reflection of A in the line BC. The lines $A'C$ and DE meet at P, and the lines $A'B$ and DF meet at Q. Prove that the lines AD, BP and CQ are concurrent (or all parallel).

Problem 2. Given positive integers m and $n \geq m$, determine the largest number of dominoes (1×2 or 2×1 rectangles) that can be placed on a rectangular board with m rows and $2n$ columns consisting of cells (1×1 squares) so that:

(i) each domino covers exactly two adjacent cells of the board;
(ii) no two dominoes overlap;
(iii) no two form a 2×2 square; and
(iv) the bottom row of the board is completely covered by n dominoes.

Problem 3. A *cubic sequence* is a sequence of integers given by $a_n = n^3 + bn^2 + cn + d$, where b, c and d are integer constants and n ranges over all integers, including negative integers.

(a) Show that there exists a cubic sequence such that the only terms of the sequence which are squares of integers are a_{2015} and a_{2016}.

(b) Determine the possible values of $a_{2015} \cdot a_{2016}$ for a cubic sequence satisfying the condition in part **(a)**.

Each of the three problems is worth 7 points.
Time allowed $4\frac{1}{2}$ hours.

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Day 2: Saturday, February 27, 2016, Bucharest

Language: English

Problem 4. Let x and y be positive real numbers such that $x + y^{2016} \geq 1$. Prove that $x^{2016} + y > 1 - 1/100$.

Problem 5. A convex hexagon $A_1B_1A_2B_2A_3B_3$ is inscribed in a circle Ω of radius R. The diagonals A_1B_2, A_2B_3, and A_3B_1 concur at X. For $i = 1, 2, 3$, let ω_i be the circle tangent to the segments XA_i and XB_i, and to the arc A_iB_i of Ω not containing other vertices of the hexagon; let r_i be the radius of ω_i.

(a) Prove that $R \geq r_1 + r_2 + r_3$.

(b) If $R = r_1 + r_2 + r_3$, prove that the six points where the circles ω_i touch the diagonals A_1B_2, A_2B_3, A_3B_1 are concyclic.

Problem 6. A set of n points in Euclidean 3-dimensional space, no four of which are coplanar, is partitioned into two subsets $\mathcal{A}$ and $\mathcal{B}$. An $\mathcal{AB}$-*tree* is a configuration of $n - 1$ segments, each of which has an endpoint in $\mathcal{A}$ and the other in $\mathcal{B}$, and such that no segments form a closed polyline. An $\mathcal{AB}$-tree is transformed into another as follows: choose three distinct segments A_1B_1, B_1A_2 and A_2B_2 in the $\mathcal{AB}$-tree such that A_1 is in $\mathcal{A}$ and $A_1B_1 + A_2B_2 > A_1B_2 + A_2B_1$, and remove the segment A_1B_1 to replace it by the segment A_1B_2. Given any $\mathcal{AB}$-tree, prove that every sequence of successive transformations comes to an end (no further transformation is possible) after finitely many steps.

Each of the three problems is worth 7 points.
Time allowed $4\frac{1}{2}$ hours.