Saturday, April 8, 2017

Problem 1. Let $ABCD$ be a convex quadrilateral with $\angle DAB = \angle BCD = 90^\circ$ and $\angle ABC > \angle CDA$. Let Q and R be points on segments BC and CD, respectively, such that line QR intersects lines AB and AD at points P and S, respectively. It is given that $PQ = RS$. Let the midpoint of BD be M and the midpoint of QR be N. Prove that the points M, N, A and C lie on a circle.

Problem 2. Find the smallest positive integer k for which there exist a colouring of the positive integers $\mathbb{Z}_{>0}$ with k colours and a function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ with the following two properties:

(i) For all positive integers m, n of the same colour, $f(m+n) = f(m) + f(n)$.

(ii) There are positive integers m, n such that $f(m+n) \neq f(m) + f(n)$.

In a colouring of $\mathbb{Z}_{>0}$ with k colours, every integer is coloured in exactly one of the k colours. In both (i) and (ii) the positive integers m, n are not necessarily different.

Problem 3. There are 2017 lines in the plane such that no three of them go through the same point. Turbo the snail sits on a point on exactly one of the lines and starts sliding along the lines in the following fashion: she moves on a given line until she reaches an intersection of two lines. At the intersection, she follows her journey on the other line turning left or right, alternating her choice at each intersection point she reaches. She can only change direction at an intersection point. Can there exist a line segment through which she passes in both directions during her journey?

Sunday, April 9, 2017

Problem 4. Let $n \geq 1$ be an integer and let $t_1 < t_2 < \ldots < t_n$ be positive integers. In a group of $t_n + 1$ people, some games of chess are played. Two people can play each other at most once. Prove that it is possible for the following two conditions to hold at the same time:

(i) The number of games played by each person is one of $t_1, t_2, \ldots, t_n$.

(ii) For every i with $1 \leq i \leq n$, there is someone who has played exactly t_i games of chess.

Problem 5. Let $n \geq 2$ be an integer. An n-tuple $(a_1, a_2, \ldots, a_n)$ of not necessarily different positive integers is *expensive* if there exists a positive integer k such that

$$(a_1 + a_2)(a_2 + a_3) \cdot \ldots \cdot (a_{n-1} + a_n)(a_n + a_1) = 2^{2k-1}.$$

a) Find all integers $n \geq 2$ for which there exists an expensive n-tuple.

b) Prove that for every odd positive integer m there exists an integer $n \geq 2$ such that m belongs to an expensive n-tuple.

There are exactly n factors in the product on the left hand side.

Problem 6. Let ABC be an acute-angled triangle in which no two sides have the same length. The reflections of the centroid G and the circumcentre O of ABC in its sides BC, CA, AB are denoted by G_1, G_2, G_3, and O_1, O_2, O_3, respectively. Show that the circumcircles of the triangles G_1G_2C, G_1G_3B, G_2G_3A, O_1O_2C, O_1O_3B, O_2O_3A and ABC have a common point.

The centroid of a triangle is the intersection point of the three medians. A median is a line connecting a vertex of the triangle to the midpoint of the opposite side.